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Simultaneous Equations

Example 1

Solve the equations by substitution. $2x + 3y = 18$ [1]

$3x - y = 5$ [2]

Example 2

Solve the equations by elimination. $2x + 3y = 18$ [1]

$3x - y = 5$ [2]

Example 3

Solve the equations and sketch the graphs. $y = x^2 + 5x - 9$ [1]

$y = 2x + 1$ [2]

Example 4

Solve the equations and sketch the graphs. $xy = 12$ [1]

$2y = 3x - 6$ [2]

Example 5

Show that the line $y = 6x - 13$ [1] is a tangent to the curve $y = x^2 + 4x - 12$ [2] and find the coordinates of the point at which the line touches the curve.

Example 6

Show that the line $x + y = 12$ [1] does not intersect the curve $y = -x^2 - 3x + 10$ [2]. Sketch the graphs.

Indices and Surds

Example 1

Simplify each of these. **a** $16^{\frac{1}{4}}$ **b** $\left(\frac{4}{9}\right)^{-\frac{3}{2}}$

Example 2

Simplify the expression $25^x \times 5^{3x-4}$.

Example 3

Find the value of x, given that: $1024 = 8^{2x} \times 16^{3x-2}$

Example 4

Find the values of p and q in the equation $\frac{32^4 \times 625^3}{8^6 \times 25^4} = 2^p 5^q$

Example 5

a Simplify $\frac{2^{3x+3} + 20(8^x)}{3(2^{3x+2})}$

b Solve the equation $\frac{3^{8x+1}}{3^{2x-5}} = \frac{243^{x+1}}{27^{x-1}}$

Example 6

Solve the following simultaneous equations.

$4^x \times 2^y = 256;$ [1] $81^x \div 27^y = 729$ [2]

Example 7

Solve the following simultaneous equations: $\frac{9^x \times 27^y}{27} = \frac{243^4}{3^{x-4} \times 3^{y-1}}$ [1]

$\frac{64^x \times 16^y}{16} = 64^y 4^x$ [2]

Example 8

Simplify $\dfrac{16^{x+1}+20\left(4^{2x}\right)}{2^{x-3}8^{x+2}}$. [4]

[Cambridge IGCSE Additional Mathematics, June 2003, P 2, Qu 4]

Example 9

Solve the equation $\dfrac{5^{4y-1}}{25^{y}}=\dfrac{125^{y+3}}{25^{2-y}}$. [4]

[Cambridge IGCSE Additional Mathematics, June 2008, P 2, Qu 8 (part)]

Example 10

a Solve the equation $9^{2x-1}=27^{x}$. [3]

b Given that $\dfrac{a^{-\frac{1}{2}}b^{\frac{2}{3}}}{\sqrt{a^{3}b^{-\frac{2}{3}}}}=a^{p}b^{q}$, find the value of p and of q. [2]

[Cambridge IGCSE Additional Mathematics, June 2009, P 1, Qu 5]

Example 11

Without using a calculator, solve, for x and y, the simultaneous equations

$$8^{x}\div 2^{y}=64,$$

$$3^{4x}\times\left(\frac{1}{9}\right)^{y-1}=81.$$ [5]

[Cambridge IGCSE Additional Mathematics, Nov 2004, P 1. Qu 3]

Example 12

a Find, in its simplest form, the product of $a^{\frac{1}{3}}+b^{\frac{2}{3}}$ and $a^{\frac{2}{3}}-a^{\frac{1}{3}}b^{\frac{2}{3}}+b^{\frac{4}{3}}$. [3]

b Given that $2^{2x+2}\times 5^{x-1}=8^{x}\times 5^{2x}$, evaluate 10^{x}. [4]

[Cambridge IGCSE Additional Mathematics, Nov 2002, P 2, Qu 9]

Example 13

(i) Express 9^{x+1} as a power of 3. [1]

(ii) Express $\sqrt[3]{27^{2x}}$ as a power of 3. [1]

(iii) Express $\dfrac{54\times\sqrt[3]{27^{2x}}}{9^{x+1}+216\left(3^{2x-1}\right)}$ as a fraction in its simplest form. [3]

[Cambridge IGCSE Additional Mathematics, Nov 2007, P 2, Qu 3]

Example 14

Solve the equation $\frac{4^x}{2^{5-x}} = \frac{2^{4x}}{8^{x-3}}$. [3]

[Cambridge IGCSE Additional Mathematics, Nov 2008, P 2, Qu 5 (part)]

Surds

The product rule	$\sqrt{a} \times \sqrt{b} = \sqrt{ab}$
The division rule	$\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$
The distributive rule (i)	$a\sqrt{c} + b\sqrt{c} = (a+b)\sqrt{c}$
The distributive rule (ii)	$a\sqrt{b} + a\sqrt{c} = a\left(\sqrt{b} + \sqrt{c}\right)$

Example 1

Calculate

a $\left(6+3\sqrt{3}\right)\left(5-2\sqrt{3}\right)$ **b** $\left(6+3\sqrt{3}\right)\left(6-3\sqrt{3}\right)$ **c** $\left(5+2\sqrt{3}\right)\left(5-2\sqrt{3}\right)$

Example 2

a $\frac{4}{\sqrt{6}} = \frac{4}{\sqrt{}}$

b $\frac{2+\sqrt{5}}{3-\sqrt{5}}$

Example 3

Express $\frac{8-3\sqrt{2}}{4+3\sqrt{2}}$ in the form $a+b\sqrt{2}$, where a and b are integers. [3]

[Cambridge IGCSE Additional Mathematics, June 2008, P 1, Qu 1]

Example 4

Given that $k = \frac{1}{\sqrt{3}}$ and that $p = \frac{1+k}{1-k}$, express in its simplest surd form

(i) p,

(ii) $p - \frac{1}{p}$. [5]

[Cambridge IGCSE Additional Mathematics, Nov 2002, P 2, Qu 3]

Example 5

Given that $\sqrt{a+b\sqrt{3}} = \frac{13}{4+\sqrt{3}}$, where a and b are integers, find, without using a calculator, the value of a and of b. [4]

[Cambridge IGCSE Additional Mathematics, Nov 2004, P 2, Qu 2]

Coordinate Geometry

Length and Midpoint formulas

Example 1

Example 3.1 *Find the length of the straight line joining the points* $A(1,3)$ *and* $B(4,7)$.

Example 2

Example 3.2 *The points P, Q and R have coordinates* $(1,11)$, $(5,7)$ *and* $(9,a)$, *respectively.*
If PQ *and* QR *are of equal length, find the possible values of* a.

Example 3

Example 3.3 *Find the coordinates of the midpoint of the line joining* $A(1,8)$ *and* $B(7,4)$.

Example 4

Example 3.4 $L(-1,-2)$ *is the midpoint of the line joining points* $P(a,-5)$ *and* $Q(3,b)$. *Find the values of* a *and* b.

Example 5

Example 3.5 $P(-1,5)$, $Q(8,10)$, $R(7,5)$ *and* S *are the vertices of the parallelogram PQRS. Calculate the coordinates of S.*

Gradient of a line

Example 6

Example 3.6 *Find the gradient of the line joining* $A(2,7)$ *and* $B(5,11)$. *Another line PQ with* $P(a,-3)$ *and* $Q(-1,5)$ *have same gradient as the line AB. Calculate the value of* a.

Example 7

Example 3.6 *Find the gradient of the line joining* $A(2,7)$ *and* $B(5,11)$. *Another line PQ with* $P(a,-3)$ *and* $Q(-1,5)$ *have same gradient as the line AB. Calculate the value of* a.

Example 8

Example 3.7 *Three points* $A(0,-5)$, $B(x,-9)$ *and* $C(3,1)$ *are collinear. Find the value of* x.

Equation of a line

Example 9

Example 3.10 *Write down the gradient and* y*-intercept of each of the following lines.*
(*i*) $y=2x+7$ (*ii*) $y=4-3x$ (*iii*) $2y=5x+1$ (*iv*) $4x-3y+2=0$

Example 10

Example 3.11 (*a*) *Find the equation of the line that passes through* $(3,7)$ *and has gradient* $\frac{4}{5}$.

(*b*) *Find the equation of the line passing through the points* $A(-4,7)$ *and* $B(3,-5)$.

Example 11

Example 3.12 (*a*) *Find the equation of the line that passes through* $(3,-2)$ *and parallel to the line* $3x+2y=6$.

(*b*) *Find the equation of the line passing through the point* $A(4,4)$ *and perpendicular to the line* $4x+5y=11$.

November 2016/12 Question 5

The line $\frac{x}{a}+\frac{y}{b}=1$, where a and b are positive constants, intersects the x- and y-axes at the points A and B respectively. The mid-point of AB lies on the line $2x+y=10$ and the distance $AB=10$. Find the values of a and b. [6]

Perpendicular bisectors

Example 12

Example 3.13 *Find the equation of the perpendicular bisector of the line joining the points* $A(4,-5)$ *and* $B(2,9)$.

Example 13

Example 3.16 *The point A* $(3,5)$ *is reflected in the line* $y = x + 1$ *to get a point B. Find the coordinates of B.*

Coordinate Geometry and Angles

Example 14

Example 3.17 *Find the acute angle between the lines* $y = 2x + 4$ *and* $y = x - 5$.

Finding the area using Coordinates

Past paper Questions

November 2016/11 Question 4

C is the mid-point of the line joining A (14, –7) to B (–6, 3). The line through C perpendicular to AB crosses the y-axis at D.

(i) Find the equation of the line CD, giving your answer in the form $y = mx + c$. [4]

(ii) Find the distance AD. [2]

November 2016/13 Question 6

Three points, A, B and C, are such that B is the mid-point of AC. The coordinates of A are (2, m) and the coordinates of B are (n, –6), where m and n are constants.

(i) Find the coordinates of C in terms of m and n. [2]

The line $y = x + 1$ passes through C and is perpendicular to AB.

(ii) Find the values of m and n. [5]

June 2016/12 Question 8

Three points have coordinates A (0, 7), B (8, 3) and C ($3k$, k). Find the value of the constant k for which

(i) C lies on the line that passes through A and B, [4]

(ii) C lies on the perpendicular bisector of AB. [4]

June 2016/13 Question 11

Triangle ABC has vertices at $A(-2, -1)$, $B(4, 6)$ and $C(6, -3)$.

(i) Show that triangle ABC is isosceles and find the exact area of this triangle. [6]

(ii) The point D is the point on AB such that CD is perpendicular to AB. Calculate the x-coordinate of D. [6]

March 2016/12 Question 5

Two points have coordinates $A(5, 7)$ and $B(9, -1)$.

(i) Find the equation of the perpendicular bisector of AB. [3]

The line through $C(1, 2)$ parallel to AB meets the perpendicular bisector of AB at the point X.

(ii) Find, by calculation, the distance BX. [5]

November 2015/12 Question 6

Points A, B and C have coordinates $A(-3, 7)$, $B(5, 1)$ and $C(-1, k)$, where k is a constant.

(i) Given that $AB = BC$, calculate the possible values of k. [3]

The perpendicular bisector of AB intersects the x-axis at D.

(ii) Calculate the coordinates of D. [5]

June 2015/11 Question 6

The line with gradient -2 passing through the point $P(3t, 2t)$ intersects the x-axis at A and the y-axis at B.

(i) Find the area of triangle AOB in terms of t. [3]

The line through P perpendicular to AB intersects the x-axis at C.

(ii) Show that the mid-point of PC lies on the line $y = x$. [4]

June 2015/12 Question 7

The point C lies on the perpendicular bisector of the line joining the points $A(4, 6)$ and $B(10, 2)$. C also lies on the line parallel to AB through $(3, 11)$.

(i) Find the equation of the perpendicular bisector of AB. [4]

(ii) Calculate the coordinates of C. [3]

June 2015/13 Question 7

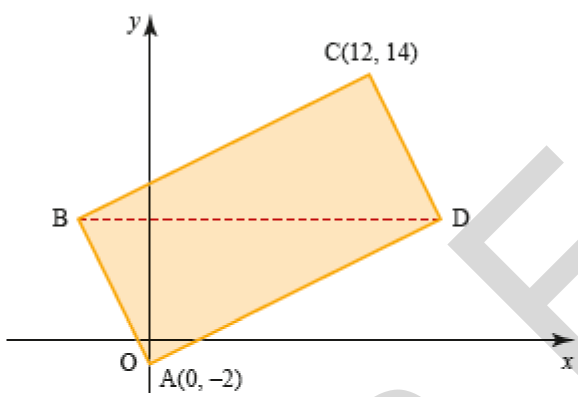
The point A has coordinates $(p, 1)$ and the point B has coordinates $(9, 3p + 1)$, where p is a constant.

(i) For the case where the distance AB is 13 units, find the possible values of p. [3]

(ii) For the case in which the line with equation $2x + 3y = 9$ is perpendicular to AB, find the value of p. [4]

November 2009/12/Q9

The diagram shows a rectangle ABCD. The point A is $(0, -2)$ and C is $(12, 14)$. The diagonal BD is parallel to the x axis.



(i) Explain why the y co-ordinate of D is 6.

The x co-ordinate of D is h.

(ii) Express the gradients of AD and CD in terms of h.

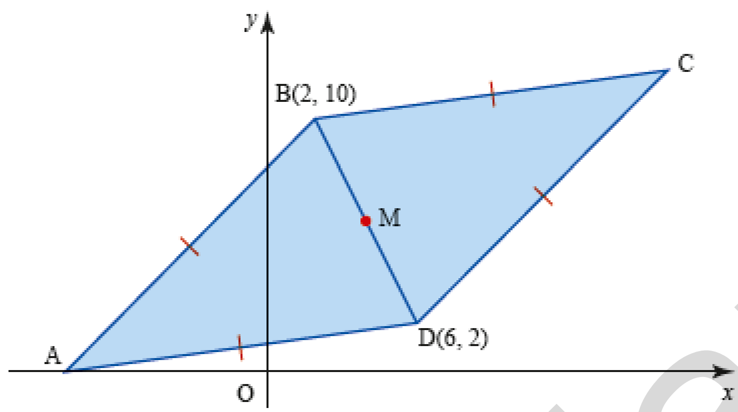
(iii) Calculate the x co-ordinates of D and B.

(iv) Calculate the area of the rectangle ABCD.

[Cambridge AS & A Level Mathematics 9709, Paper 12 Q9 November 2009]

June 2005/1/Q5

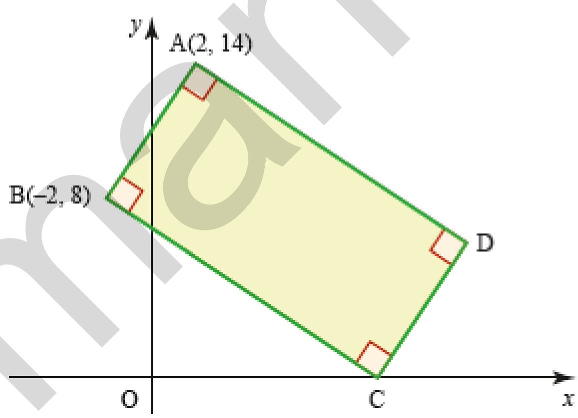
The diagram shows a rhombus ABCD. The points B and D have co-ordinates (2, 10) and (6, 2) respectively, and A lies on the x axis. The mid-point of BD is M. Find, by calculation, the co-ordinates of each of M, A and C.



[Cambridge AS & A Level Mathematics 9709, Paper 1 Q5 June 2005]

June 2007/1/Q6

The diagram shows a rectangle ABCD. The point A is (2, 14), B is (–2, 8) and C lies on the x axis.



Find

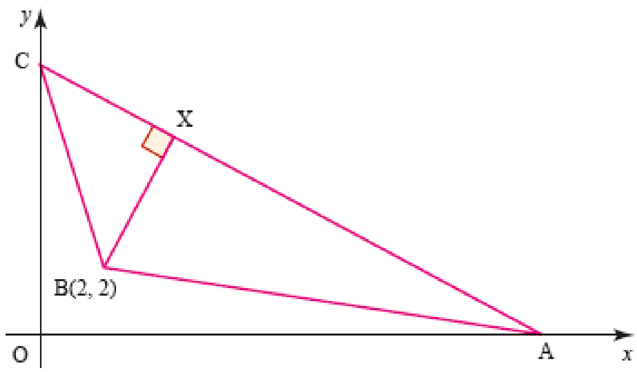
(i) the equation of BC.

(ii) the co-ordinates of C and D.

[Cambridge AS & A Level Mathematics 9709, Paper 1 Q6 June 2007]

June 2008/1/Q11

In the diagram, the points A and C lie on the x and y axes respectively and the equation of AC is $2y + x = 16$. The point B has co-ordinates (2, 2). The perpendicular from B to AC meets AC at the point X.



(i) Find the co-ordinates of X.

The point D is such that the quadrilateral ABCD has AC as a line of symmetry.

(ii) Find the co-ordinates of D.

(iii) Find, correct to 1 decimal place, the perimeter of ABCD.

[Cambridge AS & A Level Mathematics 9709, Paper 1 Q11 June 2008]

Homework: Coordinate Geometry Variant 12

1 The line L_1 has equation $2x + y = 8$. The line L_2 passes through the point $A\ (7, 4)$ and is perpendicular to L_1.

(i) Find the equation of L_2. [4]

(ii) Given that the lines L_1 and L_2 intersect at the point B, find the length of AB. [4]

Answers: (i) $2y = x + 1$; (ii) 4.47 or $\sqrt{20}$. J03/Q7

---

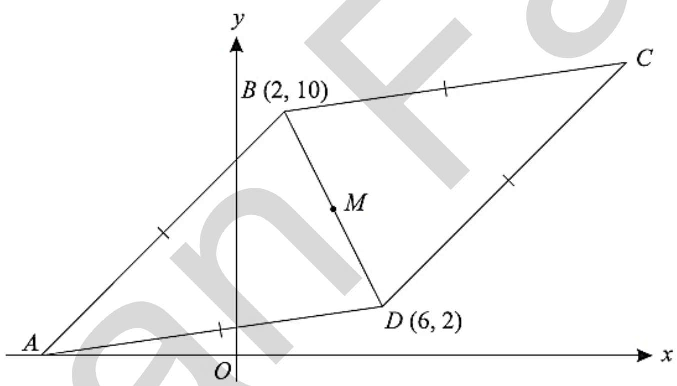
2 The curve $y = 9 - \dfrac{6}{x}$ and the line $y + x = 8$ intersect at two points. Find

(i) the coordinates of the two points, [4]

(ii) the equation of the perpendicular bisector of the line joining the two points. [4]

Answers: (i) (2, 6) and (−3, 11); (ii) $y = x + 9$. J04/Q6

3



The diagram shows a rhombus $ABCD$. The points B and D have coordinates (2, 10) and (6, 2) respectively, and A lies on the x-axis. The mid-point of BD is M. Find, by calculation, the coordinates of each of M, A and C. [6]

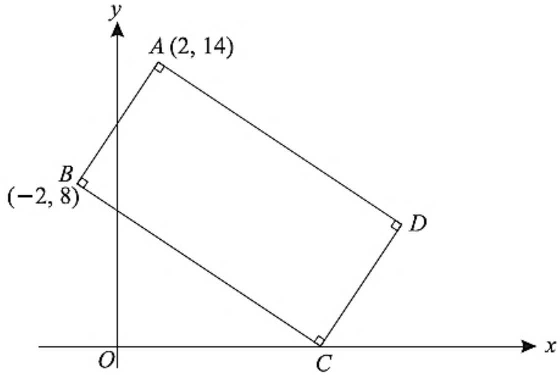
Answers: M (4, 6), A (−8, 0), C (16, 12). J05/Q5

---

4 The curve $y^2 = 12x$ intersects the line $3y = 4x + 6$ at two points. Find the distance between the two points. [6]

Answer: 3.75. J06/Q5

5



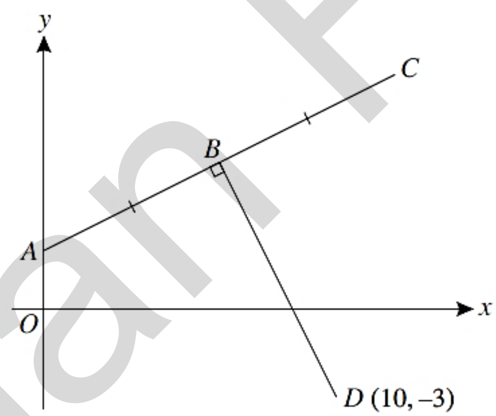
The diagram shows a rectangle $ABCD$. The point A is (2, 14), B is (−2, 8) and C lies on the x-axis. Find

(i) the equation of BC, [4]

(ii) the coordinates of C and D. [3]

Answers: (i) $3y + 2x = 20$; (ii) $C(10, 0)$, $D(14, 6)$. J07/Q6

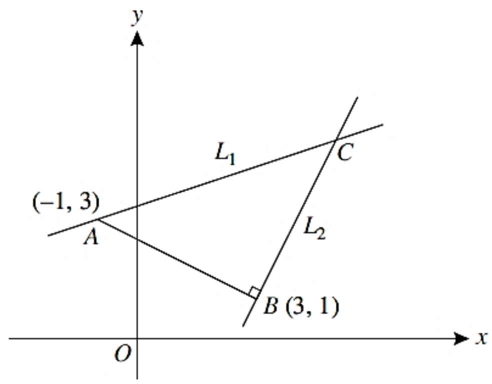
6



The diagram shows points A, B and C lying on the line $2y = x + 4$. The point A lies on the y-axis and $AB = BC$. The line from D (10, −3) to B is perpendicular to AC. Calculate the coordinates of B and C. [7]

Answer: B (6, 5), C (12, 8) J09/Q8

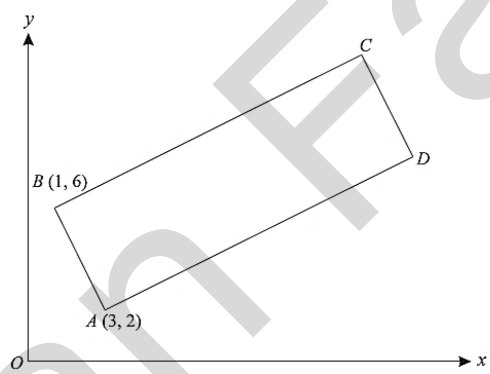
7



In the diagram, A is the point $(-1, 3)$ and B is the point $(3, 1)$. The line L_1 passes through A and is parallel to OB. The line L_2 passes through B and is perpendicular to AB. The lines L_1 and L_2 meet at C. Find the coordinates of C. [6]

Answer: (5, 5). J10/12/Q4

8



The diagram shows a rectangle $ABCD$, where A is $(3, 2)$ and B is $(1, 6)$.

(i) Find the equation of BC. [4]

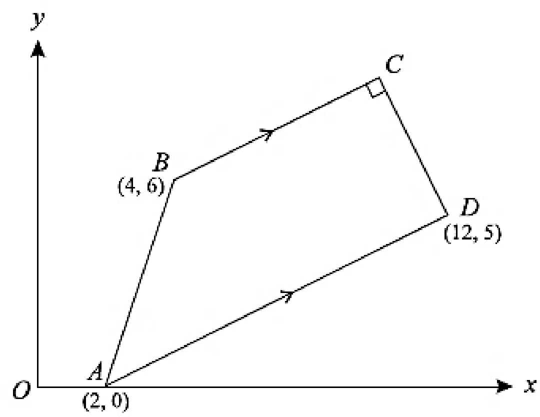
Given that the equation of AC is $y = x - 1$, find

(ii) the coordinates of C, [2]

(iii) the perimeter of the rectangle $ABCD$. [3]

Answers: (i) $2y = x+11$; (ii) $C(13, 12)$; (iii) 35.8 or $16\sqrt{5}$. N02/Q9

9



The diagram shows a trapezium $ABCD$ in which BC is parallel to AD and angle $BCD = 90°$. The coordinates of A, B and D are (2, 0), (4, 6) and (12, 5) respectively.

(i) Find the equations of BC and CD. [5]

(ii) Calculate the coordinates of C. [2]

Answers: (i) $2y = x + 8$, $y + 2x = 29$; (ii) (10, 9). **N03/Q5**

---

10 The equation of a curve is $y = x^2 - 4x + 7$ and the equation of a line is $y + 3x = 9$. The curve and the line intersect at the points A and B.

(i) The mid-point of AB is M. Show that the coordinates of M are $\left(\frac{1}{2}, 7\frac{1}{2}\right)$. [4]

(ii) Find the coordinates of the point Q on the curve at which the tangent is parallel to the line $y + 3x = 9$. [3]

(iii) Find the distance MQ. [1]

Answers: (ii) $Q(0.5, 5.25)$; (iii) 2.25. **N04/Q5**

---

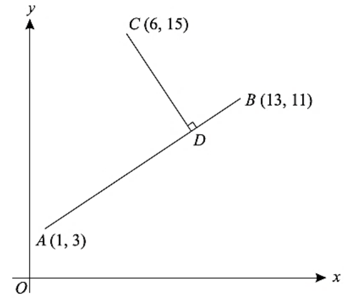
11 Three points have coordinates A (2, 6), B (8, 10) and C (6, 0). The perpendicular bisector of AB meets the line BC at D. Find

(i) the equation of the perpendicular bisector of AB in the form $ax + by = c$, [4]

(ii) the coordinates of D. [4]

Answers: (i) $3x + 2y = 31$; (ii) (7, 5). **N05/Q7**

12



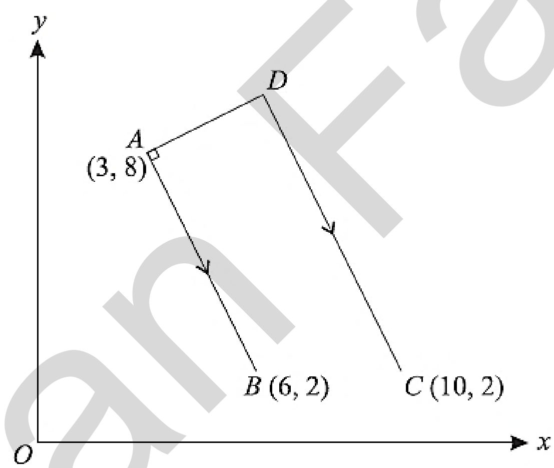
The three points A (1, 3), B (13, 11) and C (6, 15) are shown in the diagram. The perpendicular from C to AB meets AB at the point D. Find

(i) the equation of CD, [3]

(ii) the coordinates of D. [4]

Answer: (i) $2y + 3x = 48$; (ii) $D(10, 9)$. **N06/Q5**

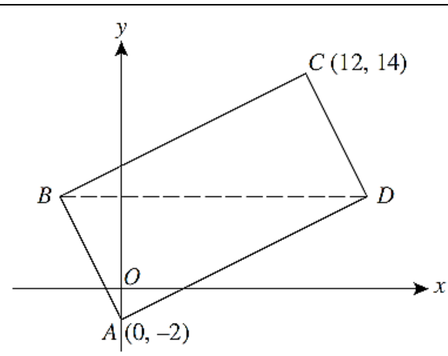
13



The three points A (3, 8), B (6, 2) and C (10, 2) are shown in the diagram. The point D is such that the line DA is perpendicular to AB and DC is parallel to AB. Calculate the coordinates of D. [7]

Answer: (6.2, 9.6). **N07/Q6**

14



The diagram shows a rectangle $ABCD$. The point A is $(0, -2)$ and C is $(12, 14)$. The diagonal BD is parallel to the x-axis.

(i) Explain why the y-coordinate of D is 6. [1]

The x-coordinate of D is h.

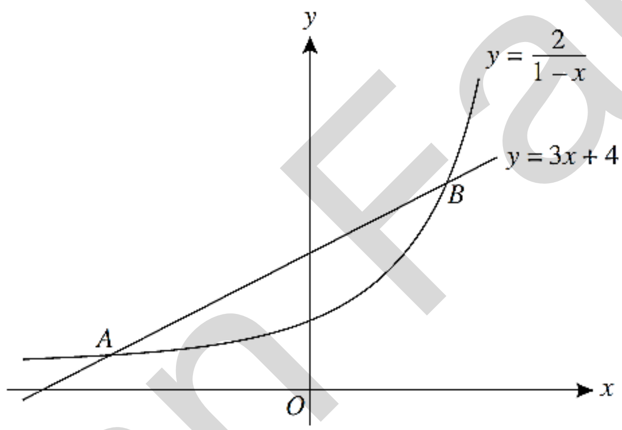
(ii) Express the gradients of AD and CD in terms of h. [3]

(iii) Calculate the x-coordinates of D and B. [4]

(iv) Calculate the area of the rectangle $ABCD$. [3]

Answers: (ii) $\frac{8}{h}$, $\frac{8}{12-h}$ or $\frac{8}{h}$, $\frac{-h}{8}$; (iii) 16, −4; (iv) 160. **N09/Q9**

15



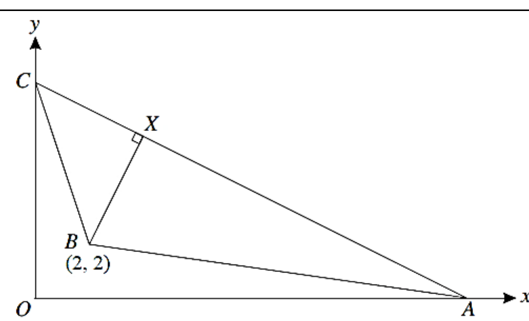
The diagram shows part of the curve $y = \frac{2}{1-x}$ and the line $y = 3x + 4$. The curve and the line meet at points A and B.

(i) Find the coordinates of A and B. [4]

(ii) Find the length of the line AB and the coordinates of the mid-point of AB. [3]

Answers: (i) $(-1, 1)$, $(\frac{2}{3}, 6)$; (ii) 5.27, $(-\frac{1}{6}, \frac{7}{2})$. **N10/12/Q8**

16



In the diagram, the points A and C lie on the x- and y-axes respectively and the equation of AC is $2y + x = 16$. The point B has coordinates $(2, 2)$. The perpendicular from B to AC meets AC at the point X.

(i) Find the coordinates of X. [4]

The point D is such that the quadrilateral $ABCD$ has AC as a line of symmetry.

(ii) Find the coordinates of D. [2]

(iii) Find, correct to 1 decimal place, the perimeter of $ABCD$. [3]

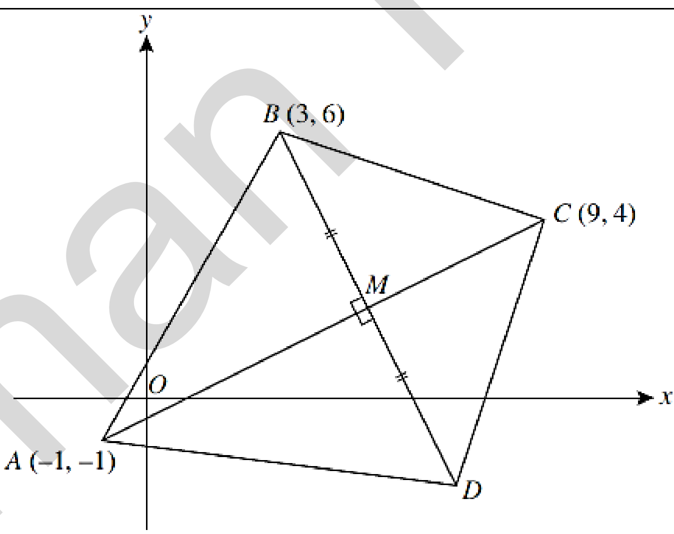
Answers: **(i)** (4, 6); **(ii)** (6, 10); **(iii)** 40.9. J08/Q11

The line L_1 passes through the points A (2, 5) and B (10, 9). The line L_2 is parallel to L_1 and passes through the origin. The point C lies on L_2 such that AC is perpendicular to L_2. Find

(i) the coordinates of C, [5]

(ii) the distance AC. [2]

Answers: **(i)** (3.6, 1.8); **(ii)** 3.58 or $\frac{8\sqrt{5}}{5}$. J11/12/Q7



The diagram shows a quadrilateral $ABCD$ in which the point A is $(-1, -1)$, the point B is $(3, 6)$ and the point C is $(9, 4)$. The diagonals AC and BD intersect at M. Angle $BMA = 90°$ and $BM = MD$. Calculate

(i) the coordinates of M and D, [7]

(ii) the ratio $AM : MC$. [2]

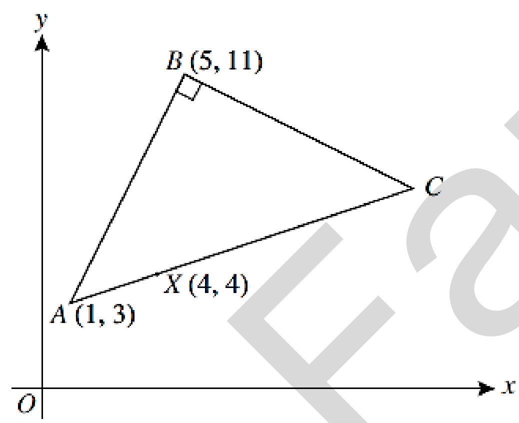
Answers: **(i)** M (5, 2), D (7, −2); **(ii)** 3 : 2 N11/12/Q9

The point A has coordinates $(-1, -5)$ and the point B has coordinates $(7, 1)$. The perpendicular bisector of AB meets the x-axis at C and the y-axis at D. Calculate the length of CD. [6]

Answer: $CD = 2.5$. **J12/12/Q4**

Find the coordinates of the point at which the perpendicular bisector of the line joining (2, 7) to (10, 3) meets the x-axis. [5]

Answer: (3½, 0) **J14/12/Q1**

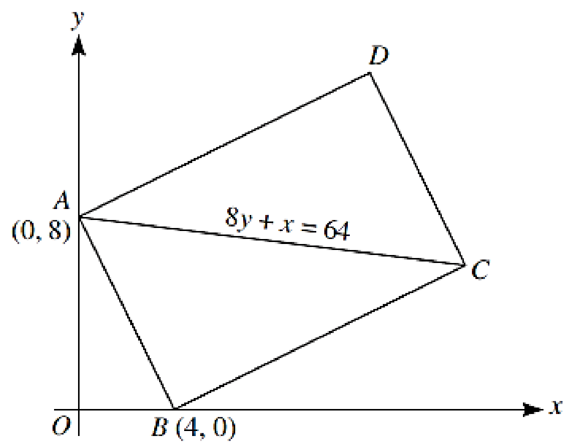


The diagram shows a triangle ABC in which A has coordinates (1, 3), B has coordinates (5, 11) and angle ABC is 90°. The point X (4, 4) lies on AC. Find

(i) the equation of BC, [3]

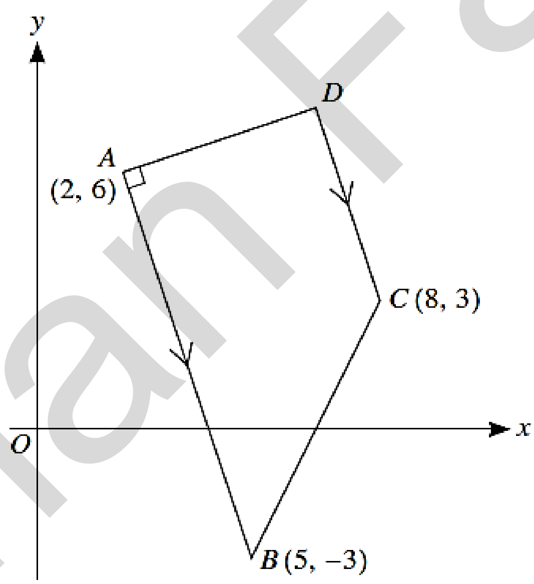
(ii) the coordinates of C. [3]

Answers: **(i)** $2y + x = 27$; **(ii)** $C(13, 7)$. **N12/12/Q5**



The diagram shows a rectangle $ABCD$ in which point A is (0, 8) and point B is (4, 0). The diagonal AC has equation $8y + x = 64$. Find, by calculation, the coordinates of C and D. [7]

Answer: **(i)** *C* (16, 6), *D* (12, 14). **N13/12/Q5**



The diagram shows a trapezium $ABCD$ in which AB is parallel to DC and angle BAD is 90°. The coordinates of A, B and C are (2, 6), (5, −3) and (8, 3) respectively.

(i) Find the equation of AD. [3]

(ii) Find, by calculation, the coordinates of D. [3]

The point E is such that $ABCE$ is a parallelogram.

(iii) Find the length of BE. [2]

Answer: **(i)** $y - 6 = \frac{1}{3}(x - 2)$; **(ii)** D (6.5, 7.5); **(iii)** 15

N14/12/Q9

Points A, B and C have coordinates $A\,(-3, 7)$, $B\,(5, 1)$ and $C\,(-1, k)$, where k is a constant.

(i) Given that $AB = BC$, calculate the possible values of k. [3]

The perpendicular bisector of AB intersects the x-axis at D.

(ii) Calculate the coordinates of D. [5]

Answers: **(i)** $k = -7$ *and* 9 **(ii)** (–2,0)

N15/12/Q6

Homework: Coordinate Geometry – Variants 11 & 13

1 The point A has coordinates $(p, 1)$ and the point B has coordinates $(9, 3p + 1)$, where p is a constant.

(i) For the case where the distance AB is 13 units, find the possible values of p. [3]

(ii) For the case in which the line with equation $2x + 3y = 9$ is perpendicular to AB, find the value of p. [4]

Answers: (i) $p = 4$ or $-\frac{11}{5}$; (ii) $p = 3$. **13/J15/7**

---

2 The line with gradient -2 passing through the point $P(3t, 2t)$ intersects the x-axis at A and the y-axis at B.

(i) Find the area of triangle AOB in terms of t. [3]

The line through P perpendicular to AB intersects the x-axis at C.

(ii) Show that the mid-point of PC lies on the line $y = x$. [4]

Answer: (i) $16t^2$ (ii) (t,t) on $y = x$ **11/J15/6**

---

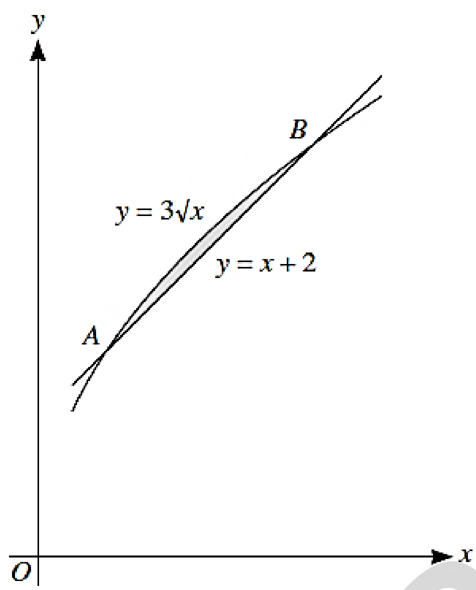
3 A is the point $(a, 2a - 1)$ and B is the point $(2a + 4, 3a + 9)$, where a is a constant.

(i) Find, in terms of a, the gradient of a line perpendicular to AB. [3]

(ii) Given that the distance AB is $\sqrt{(260)}$, find the possible values of a. [4]

Answers: (i) $\frac{-(a+4)}{a+10}$; (ii) 4, -18 **13/N14/6**

4



The diagram shows parts of the graphs of $y = x + 2$ and $y = 3\sqrt{x}$ intersecting at points A and B.

(i) Write down an equation satisfied by the x-coordinates of A and B. Solve this equation and hence find the coordinates of A and B. [4]

Answers: **(i)** (1, 3), (4, 6); 13/N14/9

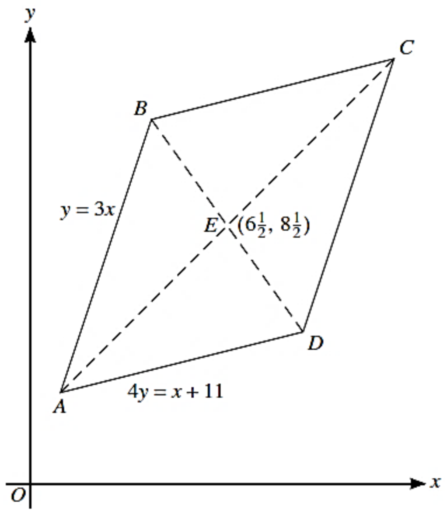
5 The line $4x + ky = 20$ passes through the points A (8, −4) and B (b, $2b$), where k and b are constants.

(i) Find the values of k and b. [4]

(ii) Find the coordinates of the mid-point of AB. [1]

Answers: **(i)** $k = 3$, $b = 2$ **(ii)** (5, 0) 11/N14/4

6



The diagram shows a parallelogram $ABCD$, in which the equation of AB is $y = 3x$ and the equation of AD is $4y = x + 11$. The diagonals AC and BD meet at the point $E\left(6\frac{1}{2}, 8\frac{1}{2}\right)$. Find, by calculation, the coordinates of A, B, C and D. [9]

Answers: A(1, 3), B(4, 12), C(12, 14), D(9, 5). 13/J14/11

7 The coordinates of points A and B are $(a, 2)$ and $(3, b)$ respectively, where a and b are constants. The distance AB is $\sqrt{(125)}$ units and the gradient of the line AB is 2. Find the possible values of a and of b. [6]

Answer: $a = -2$ or 8, $b = 12$ or -8. 11/J14/7

8 The point A has coordinates (3, 1) and the point B has coordinates (−21, 11). The point C is the mid-point of AB.

(i) Find the equation of the line through A that is perpendicular to $y = 2x - 7$. [2]

(ii) Find the distance AC. [3]

Answer: $y - 1 = -\frac{1}{2}(x - 3)$, $AC = 13$ 13/N13/3

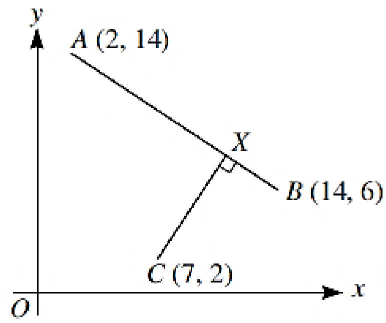
9 The point A has coordinates (−1, 6) and the point B has coordinates (7, 2).

(i) Find the equation of the perpendicular bisector of AB, giving your answer in the form $y = mx + c$. [4]

(ii) A point C on the perpendicular bisector has coordinates (p, q). The distance OC is 2 units, where O is the origin. Write down two equations involving p and q and hence find the coordinates of the possible positions of C. [5]

Answers: **(i)** $y = 2x - 2$; **(ii)** $(0, -2)$, $(8/5, 6/5)$. **11/N13/7**

10



The diagram shows three points $A\,(2, 14)$, $B\,(14, 6)$ and $C\,(7, 2)$. The point X lies on AB, and CX is perpendicular to AB. Find, by calculation,

(i) the coordinates of X, [6]

(ii) the ratio $AX : XB$. [2]

Answers: **(i)** $(11, 8)$; **(ii)** 3:1. **13/J13/7**

---

11 A curve has equation $y = x^2 - 4x + 4$ and a line has equation $y = mx$, where m is a constant.

(i) For the case where $m = 1$, the curve and the line intersect at the points A and B. Find the coordinates of the mid-point of AB. [4]

Answers: **(i)** $(2\frac{1}{2}, 2\frac{1}{2})$; **11/J13/7**

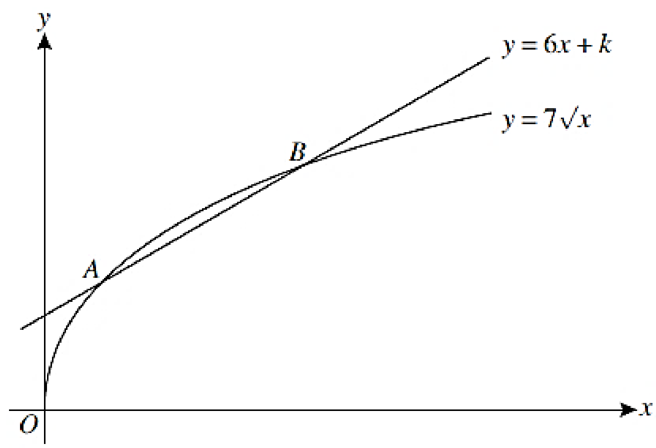
---

12 The equation of a line is $2y + x = k$, where k is a constant, and the equation of a curve is $xy = 6$.

(i) In the case where $k = 8$, the line intersects the curve at the points A and B. Find the equation of the perpendicular bisector of the line AB. [6]

Answers: **(i)** $y = 2x - 6$ **13/J12/10**

13



The diagram shows the curve $y = 7\sqrt{x}$ and the line $y = 6x + k$, where k is a constant. The curve and the line intersect at the points A and B.

(i) For the case where $k = 2$, find the x-coordinates of A and B. [4]

Answers: **(i)** $\frac{4}{9}$ or $\frac{1}{4}$ **11/J12/5**

---

14 The coordinates of A are $(-3, 2)$ and the coordinates of C are $(5, 6)$. The mid-point of AC is M and the perpendicular bisector of AC cuts the x-axis at B.

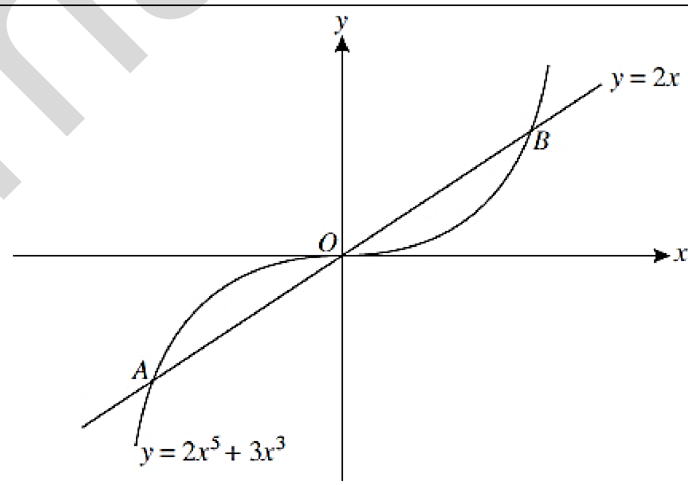
(i) Find the equation of MB and the coordinates of B. [5]

(ii) Show that AB is perpendicular to BC. [2]

(iii) Given that $ABCD$ is a square, find the coordinates of D and the length of AD. [2]

Answers: **(i)** $y = -2x + 6$, $B = (3, 0)$ **(ii)** $m_1 = -1/3, m_2 = 3$, $m_1 m_2 = -1$ **(iii)** $D = (-1, 8)$, $|AD| = \sqrt{40}$ **11/J12/9**

15



The diagram shows the curve $y = 2x^5 + 3x^3$ and the line $y = 2x$ intersecting at points A, O and B.

(i) Show that the x-coordinates of A and B satisfy the equation $2x^4 + 3x^2 - 2 = 0$. [2]

(ii) Solve the equation $2x^4 + 3x^2 - 2 = 0$ and hence find the coordinates of A and B, giving your answers in an exact form. [3]

Answers: (ii) $(1/\sqrt{2}, 2/\sqrt{2})$, $(-1/\sqrt{2}, -2/\sqrt{2})$ 13/N11/3

16 A line has equation $y = kx + 6$ and a curve has equation $y = x^2 + 3x + 2k$, where k is a constant.

(i) For the case where $k = 2$, the line and the curve intersect at points A and B. Find the distance AB and the coordinates of the mid-point of AB. [5]

Answers: (i) $\sqrt{45}$, $(-\frac{1}{2}, 5)$; 11/N11/9

17 The line $\frac{x}{a} + \frac{y}{b} = 1$, where a and b are positive constants, meets the x-axis at P and the y-axis at Q. Given that $PQ = \sqrt{(45)}$ and that the gradient of the line PQ is $-\frac{1}{2}$, find the values of a and b. [5]

Answers: 6, 3. 13/J11/3

18 The line $x - y + 4 = 0$ intersects the curve $y = 2x^2 - 4x + 1$ at points P and Q. It is given that the coordinates of P are (3, 7).

(ii) Find the coordinates of Q. [3]

(iii) Find the equation of the line joining Q to the mid-point of AP. [3]

(ii) (–0.5, 3.5); (iii) $5y + x = 17$. 11/J11/10

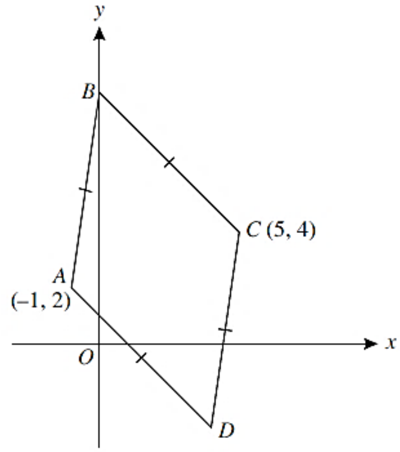
19 Points A, B and C have coordinates (2, 5), (5, –1) and (8, 6) respectively.

(i) Find the coordinates of the mid-point of AB. [1]

(ii) Find the equation of the line through C perpendicular to AB. Give your answer in the form $ax + by + c = 0$. [3]

Answers: (i) (3.5, 2); (ii) $x - 2y + 4 = 0$. 13/N10/2

20



The diagram shows a rhombus $ABCD$ in which the point A is $(-1, 2)$, the point C is $(5, 4)$ and the point B lies on the y-axis. Find

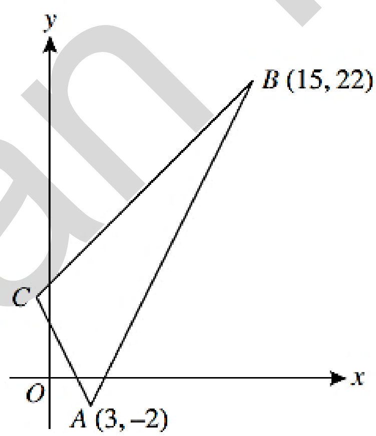
(i) the equation of the perpendicular bisector of AC, [3]

(ii) the coordinates of B and D, [3]

(iii) the area of the rhombus. [3]

Answers: **(i)** $y + 3x = 9$; **(ii)** (0,9), (4,–3); **(iii)** 40. **13/J10/8**

21



The diagram shows a triangle ABC in which A is $(3, -2)$ and B is $(15, 22)$. The gradients of AB, AC and BC are $2m$, $-2m$ and m respectively, where m is a positive constant.

(i) Find the gradient of AB and deduce the value of m. [2]

(ii) Find the coordinates of C. [4]

The perpendicular bisector of AB meets BC at D.

(iii) Find the coordinates of D. [4]

Answers: **(i)** 1; **(ii)** (–1, 6); **(iii)** (5, 12). **11/J10/8**

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22 Triangle ABC has vertices at $A\,(-2, -1)$, $B\,(4, 6)$ and $C\,(6, -3)$.

(i) Show that triangle ABC is isosceles and find the exact area of this triangle. [6]

(ii) The point D is the point on AB such that CD is perpendicular to AB. Calculate the x-coordinate of D. [6]

Answers: **(i)** 34; **(ii)** 2/5. **J16/13/Q11**

---

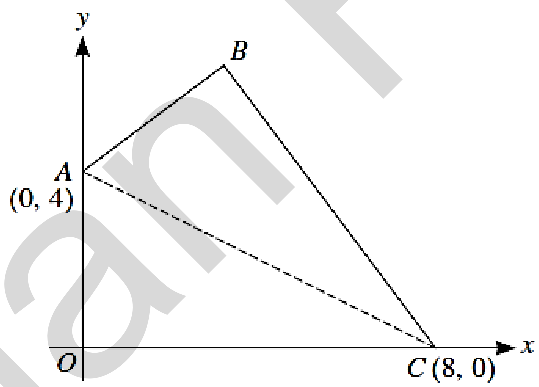
23 $A\,(-1, 1)$ and $P\,(a, b)$ are two points, where a and b are constants. The gradient of AP is 2.

(i) Find an expression for b in terms of a. [2]

(ii) $B\,(10, -1)$ is a third point such that $AP = AB$. Calculate the coordinates of the possible positions of P. [6]

Answers : **(i)** $b = 2a + 3$; **(ii)** (4, 11), (–6, –9). **J17/13/Q8**

24



The diagram shows a kite $OABC$ in which AC is the line of symmetry. The coordinates of A and C are (0, 4) and (8, 0) respectively and O is the origin.

(i) Find the equations of AC and OB. [4]

(ii) Find, by calculation, the coordinates of B. [3]

Answers: **(i)** $AC; y = \frac{-x}{2} + 4$, $OB; y = 2x$ **(ii)** $(3.2, 6.4)$ **J18/11/Q5**

25 The coordinates of points A and B are $(-3k - 1, k + 3)$ and $(k + 3, 3k + 5)$ respectively, where k is a constant $(k \neq -1)$.

(i) Find and simplify the gradient of AB, showing that it is independent of k. [2]

(ii) Find and simplify the equation of the perpendicular bisector of AB. [5]

Answers: **(i)** $\frac{1}{2}$; **(ii)** $y + 2x = 6$. **J18/13/Q6**

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26 C is the mid-point of the line joining A (14, −7) to B (−6, 3). The line through C perpendicular to AB crosses the y-axis at D.

(i) Find the equation of the line CD, giving your answer in the form $y = mx + c$. [4]

(ii) Find the distance AD. [2]

Answers: **(i)** $y = 2x - 10$ **(ii)** $\sqrt{205}$ **N16/11/Q4**

---

27 Three points, A, B and C, are such that B is the mid-point of AC. The coordinates of A are $(2, m)$ and the coordinates of B are $(n, -6)$, where m and n are constants.

(i) Find the coordinates of C in terms of m and n. [2]

The line $y = x + 1$ passes through C and is perpendicular to AB.

(ii) Find the values of m and n. [5]

Answers: **(i)** $(2n - 2, -12 - m)$; **(ii)** $m = -9$, $n = -1$. **N16/13/Q6**

---

28 Two points A and B have coordinates $(3a, -a)$ and $(-a, 2a)$ respectively, where a is a positive constant.

(i) Find the equation of the line through the origin parallel to AB. [2]

(ii) The length of the line AB is $3\frac{1}{3}$ units. Find the value of a. [3]

Answers: **(i)** $y = -\frac{3x}{4}$, **(ii)** $\frac{2}{3}$ **N18/11/Q3**

---

29 Two points A and B have coordinates $(-1, 1)$ and $(3, 4)$ respectively. The line BC is perpendicular to AB and intersects the x-axis at C.

(i) Find the equation of BC and the x-coordinate of C. [4]

(ii) Find the distance AC, giving your answer correct to 3 decimal places. [2]

Answers: **(i)** $y=\frac{-4x}{3}+8$ $x=6$; **(ii)** 7.071

N18/13/Q4

Quadratics

Solving Quadratic Equations

Example 1

Solve **a)** $6x^2 + 11x - 35 = 0$

b) $20x^2 + 80 = 82x$.

Disguised Quadratic Equations

Example 2

Solve the equation $x - \dfrac{2}{x} = 2.$

Example 3

Solve the equation $x - \sqrt{25 - x^2} = 1.$

Example 4

Solve the equation $x^4 + 3x^2 - 4 = 0.$

Example 5

Solve the equation $2x^6 - 3x^3 = 8$. Write your answer correct to 2 decimal places.

Example 6

Solve the equation $2^{2x} - 3(2^{x+2}) + 32 = 0.$

Sketching a Quadratic Curve

Example 7

Example 1.4 *For each of the following quadratic polynomials, write down the coordinates of the turning point, stating whether it is minimum or maximum. Determine the line of symmetry and hence sketch the curve.*

$(i)\ y = 2x^2 - 12x + 7 \quad (ii)\ y = 3 - 4x - 2x^2 \quad (iii)\ y = (x-1)(x-2)$

Example 8

Example 1.5 *The curve* $y = 2x^2 + bx + c$ *has a minimum point at* $\left(-\dfrac{1}{2}, -\dfrac{11}{2}\right)$. *Find the values of* b *and* c.

The Completed Square form

Example 9

Express $x^2 + 10x - 3$ in the form $(x + p)^2 + q$, where p and q are constants.

Example 10

Express $2x^2 - 12x + 1$ in the form $a(x + p)^2 + q$, where a, p and q are constants.

Example 11

Express $3 + 4x - x^2$ in the form $q - (x + p)^2$, where p and q are constants.

Example 12

Express $5 - x - 2x^2$ in the form $a - b(x + c)^2$ and hence or otherwise find its maximum value and the value of x where this occurs.

Solving Quadratic Inequalities

Example 13

Solve the inequality $x^2 - 3x - 4 \geq 0$.

Example 14

Find the range of values of x for which $x^2 + 2 < 3x$

Example 15

Solve the inequality $6 + x - x^2 > 0$.

Example 16

Solve the inequality $(x + 5)(2x + 1) \leq 0$.

Nature of roots

Work out whether each of these quadratic equations has two distinct roots, equal roots or no real roots.

a) $x^2 - 3x + 5 = 0$

b) $3x^2 + x - 6 = 0$

c) $25x^2 + 20x + 4 = 0$

Example 17

Example 1.10 *Find the values of p for which the quadratic equation $x^2 + (2 + p)x + (13 - p) = 0$ has equal roots. For these values of p, find the roots.*

Example 18

The equation $qx^2 - 2(q+3)x + q - 1 = 0$ has two different real roots. What is the range of values of q?

Example 19

Find the range of values of k for which the equation $kx^2 + 3x + k = 2kx + 5$ has no real roots.

Example 20

Find the range of values of t if the equation $3x^2 - 3tx + (t^2 - t - 3) = 0$ has two different real roots

Example 21

If the equation $qx(x-1) + q + 3 = 0$ has no real roots, find the range of values of q?

Example 22

Example 1.18 *Find the range of values of* p *for which the equation* $px^2 + 4x + (p+3) = 0$ *has real roots.*

Intersection of a line and curve

Example 23

Solve simultaneously $y^2 + xy + 4x = 7$ and $x - y = 3$.

of simultaneous equations where we do not have one linear equation and one quadratic equation.

Example 23

For what values of k will the x-axis be a tangent to the curve $y = kx^2 + (1 + k)x + k$?

Example 24

Determine the set of values of the constant k for which the line $y = 4x + k$ does not intersect the curve $y = x^2$.

[Cambridge AS & A Level Mathematics 9709, Paper 1 Q1 November 2007]

Example 25

Find the set of values of k for which the line $y + 4 = kx$ intersects the curve $y = x^2$ at two distinct points.

Example 26

Find the range of values of p for which the line $2x - y = p$ ***meets the curve*** $x(x - y) = 4$.

Example 25

Example 1.19 *A curve has equation* $x^2 + xy + 2 = 0$ *and a line has equation* $y = x + p$, *where* p *is a constant.*

(i) Find the set of values of p *for which the curve and the line have no common points.*

(ii) State the values of p *for which the line is a tangent to the curve and find the coordinates of the points where the line touches the curve.*

Homework: Quadratics Variant 12

1 (i) Express $2x^2 + 8x - 10$ in the form $a(x+b)^2 + c$. [3]

(ii) For the curve $y = 2x^2 + 8x - 10$, state the least value of y and the corresponding value of x. [2]

(iii) Find the set of values of x for which $y \geqslant 14$. [3]

Answers: (i) $a = 2, b = 2, c = -18$; (ii) $x = -2,\ y = -18$; (iii) $x \geqslant 2, x \leqslant -6$; (iv) -2; (v) $f^{-1}(x) = \sqrt{\frac{x+18}{2}} - 2$. **N02/Q11**

2 The equation of a curve is $y = 8x - x^2$.

(i) Express $8x - x^2$ in the form $a - (x+b)^2$, stating the numerical values of a and b. [3]

(ii) Hence, or otherwise, find the coordinates of the stationary point of the curve. [2]

(iii) Find the set of values of x for which $y \geqslant -20$. [3]

Answers: (i) $16 - (x-4)^2$, $a = 16$, $b = -4$; (ii) $(4, 16)$; (iii) $-2 \leqslant x \leqslant 10$; **J03/Q11**

3 The equation of a curve is $xy = 12$ and the equation of a line l is $2x + y = k$, where k is a constant.

(i) In the case where $k = 11$, find the coordinates of the points of intersection of l and the curve. [3]

(ii) Find the set of values of k for which l does not intersect the curve. [4]

(iii) In the case where $k = 10$, one of the points of intersection is $P(2, 6)$. Find the angle, in degrees correct to 1 decimal place, between l and the tangent to the curve at P. [4]

Answers: (i) $(1\frac{1}{2}, 8)$, $(4, 3)$; (ii) $-\sqrt{96} < k < \sqrt{96}$; (iii) $8.1°$. **N05/Q9**

4 Find the value of the constant c for which the line $y = 2x + c$ is a tangent to the curve $y^2 = 4x$. [4]

Answer: $\frac{1}{2}$. **J07/Q1**

5 Find the real roots of the equation $\frac{18}{x^4} + \frac{1}{x^2} = 4$. [4]

Answer: $x = \pm 1.5$. **J07/Q4**

6 Determine the set of values of the constant k for which the line $y = 4x + k$ does not intersect the curve $y = x^2$. [3]

Answer: $k < -4$. N07/Q1

---

7 The equation of a curve C is $y = 2x^2 - 8x + 9$ and the equation of a line L is $x + y = 3$.

(i) Find the x-coordinates of the points of intersection of L and C. [4]

Answers: (i) 2, $1\frac{1}{2}$. J08/Q4

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8 Find the set of values of k for which the line $y = kx - 4$ intersects the curve $y = x^2 - 2x$ at two distinct points. [4]

Answer: $k > 2$ or $k < -6$ J09/Q2

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8 (ii) Determine the set of values of k for which the line $2y = x + k$ does not intersect the curve $y = x^2 - 4x + 7$. [4]

(ii) $k < 3.875$. N09/Q10ii

---

9 A curve has equation $y = kx^2 + 1$ and a line has equation $y = kx$, where k is a non-zero constant.

(i) Find the set of values of k for which the curve and the line have no common points. [3]

(ii) State the value of k for which the line is a tangent to the curve and, for this case, find the coordinates of the point where the line touches the curve. [4]

Answers: (i) $0 < k < 4$; (ii) 4, $(\frac{1}{2}, 2)$. N10/Q6

---

10 Find the coordinates of the points of intersection of the line $y + 2x = 11$ and the curve $xy = 12$. [4]

Answers: (1.5, 8) and (4, 3). N03/Q1

---

11 Find the value of the constant c for which the line $y = 2x + c$ is a tangent to the curve $y^2 = 4x$. [4]

Answer: $\frac{1}{2}$. J07/Q1

12 The equation $x^2 + px + q = 0$, where p and q are constants, has roots -3 and 5.

(i) Find the values of p and q. [2]

(ii) Using these values of p and q, find the value of the constant r for which the equation $x^2 + px + q + r = 0$ has equal roots. [3]

Answers: **(i)** -2, -15; **(ii)** 16. **J11/12/Q3**

---

13 The equation of a curve is $y^2 + 2x = 13$ and the equation of a line is $2y + x = k$, where k is a constant.

(i) In the case where $k = 8$, find the coordinates of the points of intersection of the line and the curve. [4]

(ii) Find the value of k for which the line is a tangent to the curve. [3]

Answers: **(i)** (2, 3), (6, 1); **(ii)** 8½ **N11/12/Q4**

---

14 The straight line $y = mx + 14$ is a tangent to the curve $y = \frac{12}{x} + 2$ at the point P. Find the value of the constant m and the coordinates of P. [5]

Answer: $m = -8$. $P(2, 8)$. **J13/12/Q3**

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15 The line $y = \frac{x}{k} + k$, where k is a constant, is a tangent to the curve $4y = x^2$ at the point P. Find

(i) the value of k, [3]

(ii) the coordinates of P. [3]

Answers: **(i)** $k = -1$; **(ii)** $P(-2, 1)$. **N12/12/Q4**

Homework: Quadratics – Variants 11 & 13

1 Express $2x^2 - 12x + 7$ in the form $a(x+b)^2 + c$, where a, b and c are constants. [3]

Answer: $2(x-3)^2 - 11$. J15/13/1

2 (i) Express $9x^2 - 12x + 5$ in the form $(ax+b)^2 + c$. [3]

(ii) Determine whether $3x^3 - 6x^2 + 5x - 12$ is an increasing function, a decreasing function or neither. [3]

Answers: **(i)** $(3x-2)^2 + 1$; **(ii)** Increasing since derivative = $(3x-2)^2 + 1$ which is greater than 0 13/N14/3

3 Find the set of values of k for which the line $y = 2x - k$ meets the curve $y = x^2 + kx - 2$ at two distinct points. [5]

Answer: $k < 2$, $k > 6$ 11/N14/5

4 (i) Express $2x^2 - 10x + 8$ in the form $a(x+b)^2 + c$, where a, b and c are constants, and use your answer to state the minimum value of $2x^2 - 10x + 8$. [4]

(ii) Find the set of values of k for which the equation $2x^2 - 10x + 8 = kx$ has no real roots. [4]

Answers: **(i)** $2(x - 2\frac{1}{2})^2 - 4\frac{1}{2}$, minimum value $-4½$; **(ii)** $-18 < k < -2$. 13/J14/8

5 (i) Express $4x^2 - 12x$ in the form $(2x+a)^2 + b$. [2]

(ii) Hence, or otherwise, find the set of values of x satisfying $4x^2 - 12x > 7$. [2]

Answers: **(i)** $(2x-3)^2 - 9$; **(ii)** $x < -½$, $x > 3½$. 11/J14/2

6 A line has equation $y = 2x + c$ and a curve has equation $y = 8 - 2x - x^2$.

(i) For the case where the line is a tangent to the curve, find the value of the constant c. [3]

Answers: **(i)** $c = 12$; **(ii)** 1⅓. 11/J14/11

7 Solve the inequality $x^2 - x - 2 > 0$. [3]

Answer: $x < -1$, $x > 2$. 13/N13/1

8 It is given that $f(x) = (2x-5)^3 + x$, for $x \in \mathbb{R}$. Show that f is an increasing function. [3]

Answer: $f'(x) = 6(2x-5)^2 + 1$. This is > 0 for all values of x and hence the function is increasing. 11/J13/1

---

8 A curve has equation $y = x^2 - 4x + 4$ and a line has equation $y = mx$, where m is a constant.

(i) For the case where $m = 1$, the curve and the line intersect at the points A and B. Find the coordinates of the mid-point of AB. [4]

(ii) Find the non-zero value of m for which the line is a tangent to the curve, and find the coordinates of the point where the tangent touches the curve. [5]

Answers: **(i)** $(2\frac{1}{2}, 2\frac{1}{2})$; **(ii)** $m = -8$, $(-2, 16)$. 11/J13/7

---

9 A straight line has equation $y = -2x + k$, where k is a constant, and a curve has equation $y = \frac{2}{x-3}$.

(i) Show that the x-coordinates of any points of intersection of the line and curve are given by the equation $2x^2 - (6+k)x + (2+3k) = 0$. [1]

(ii) Find the two values of k for which the line is a tangent to the curve. [3]

The two tangents, given by the values of k found in part **(ii)**, touch the curve at points A and B.

(iii) Find the coordinates of A and B and the equation of the line AB. [6]

Answers: **(i)** 2, 10; **(ii)** $(4, 2)$, $(2, -2)$, $y - 2 = 2(x-4)$. 13/N12/10

---

10 The equation of a line is $2y + x = k$, where k is a constant, and the equation of a curve is $xy = 6$.

(i) In the case where $k = 8$, the line intersects the curve at the points A and B. Find the equation of the perpendicular bisector of the line AB. [6]

(ii) Find the set of values of k for which the line $2y + x = k$ intersects the curve $xy = 6$ at two distinct points. [3]

Answers: **(i)** $y = 2x - 6$ **(ii)** $k < -\sqrt{48}$ and $k > \sqrt{48}$ 13/J12/10

---

11 **(ii)** Find the value of k for which $y = 6x + k$ is a tangent to the curve $y = 7\sqrt{x}$. [2]

11/J12/5

(ii) $\frac{49}{24}$

12 **(i)** A straight line passes through the point (2, 0) and has gradient m. Write down the equation of the line. [1]

(ii) Find the two values of m for which the line is a tangent to the curve $y = x^2 - 4x + 5$. For each value of m, find the coordinates of the point where the line touches the curve. [6]

(iii) Express $x^2 - 4x + 5$ in the form $(x + a)^2 + b$ and hence, or otherwise, write down the coordinates of the minimum point on the curve. [2]

Answers: **(i)** $y=m(x-2)$, **(ii)** (3, 2), (1, 2), **(iii)** $(x-2)^2+1$, giving minimum at (2, 1)

13/N11/7

---

13 A line has equation $y = kx + 6$ and a curve has equation $y = x^2 + 3x + 2k$, where k is a constant.

(i) For the case where $k = 2$, the line and the curve intersect at points A and B. Find the distance AB and the coordinates of the mid-point of AB. [5]

(ii) Find the two values of k for which the line is a tangent to the curve. [4]

11/N11/9

Answers: **(i)** $\sqrt{45}$, $(-\frac{1}{2}, 5)$; **(ii)** 3, 11.

---

14 Find the set of values of m for which the line $y = mx + 4$ intersects the curve $y = 3x^2 - 4x + 7$ at two distinct points. [5]

13/J11/2

Answer: $m > 2$, $m < -10$.

---

15 **(i)** Express $2x^2 - 4x + 1$ in the form $a(x + b)^2 + c$ and hence state the coordinates of the minimum point, A, on the curve $y = 2x^2 - 4x + 1$. [4]

11/J11/10

Answers: **(i)** $2(x-1)^2-1$, $(1, -1)$;

---

16 A curve has equation $y = x^2 - x + 3$ and a line has equation $y = 3x + a$, where a is a constant.

(i) Show that the x-coordinates of the points of intersection of the line and the curve are given by the equation $x^2 - 4x + (3 - a) = 0$. [1]

(ii) For the case where the line intersects the curve at two points, it is given that the x-coordinate of one of the points of intersection is -1. Find the x-coordinate of the other point of intersection. [2]

(iii) For the case where the line is a tangent to the curve at a point P, find the value of a and the coordinates of P. [4]

N15/11/Q6

17 A line has equation $y = 2x - 7$ and a curve has equation $y = x^2 - 4x + c$, where c is a constant. Find the set of possible values of c for which the line does not intersect the curve. [3]

N15/13/Q1

18 Find the coordinates of the points of intersection of the curve $y = x^{\frac{2}{3}} - 1$ with the curve $y = x^{\frac{1}{3}} + 1$. [4]

Answer : (8, 3), (–1, 0). J17/13/Q3

19 Express $3x^2 - 12x + 7$ in the form $a(x + b)^2 + c$, where a, b and c are constants. [3]

Answer: $3(x - 2)^2 - 5$. J18/13/Q1

20 (i) Express $x^2 + 6x + 2$ in the form $(x + a)^2 + b$, where a and b are constants. [2]

(ii) Hence, or otherwise, find the set of values of x for which $x^2 + 6x + 2 > 9$. [2]

Answers: **(i)** $(x + 3)^2 - 7$ **(ii)** $x < -7, x > 1$ N16/11/Q1

21 Find the set of values of k for which the curve $y = kx^2 - 3x$ and the line $y = x - k$ do not meet. [3]

Answer: $k > 2$, $k < -2$. N16/13/Q1

22 Find the set of values of a for which the curve $y = -\frac{2}{x}$ and the straight line $y = ax + 3a$ meet at two distinct points. [4]

N17/13/Q2

Answer: $a < 0$, $a > \frac{8}{9}$

23 Showing all necessary working, solve the equation $4x - 11x^{\frac{1}{2}} + 6 = 0$. [3]

Answers: $\frac{9}{16}, 4$ N18/11/Q1

24 A line has equation $y = x + 1$ and a curve has equation $y = x^2 + bx + 5$. Find the set of values of the constant b for which the line meets the curve. [4]

Answer: $b \leqslant -3$, $b \geqslant 5$ N18/11/Q2

25 A curve has equation $y = 2x^2 - 3x + 1$ and a line has equation $y = kx + k^2$, where k is a constant.

(i) Show that, for all values of k, the curve and the line meet. [4]

(ii) State the value of k for which the line is a tangent to the curve and find the coordinates of the point where the line touches the curve. [4]

N18/13/Q9

Answer: **(ii)** $k = -\frac{1}{3}$, $\left(\frac{2}{3}, -\frac{1}{9}\right)$

Equations of Circles

Example 1

Find the equation of the circle with centre $(-5, 7)$ and radius 6.

Example 2

Find the centre and radius of the circle with equation $x^2 + y^2 - 10x + 12y + 12 = 0$.

Example 3

Find the equation of the circle with centre $(4, -3)$ that passes through the point $(-2, 5)$.

Example 4

Determine the equation of the tangent to the circle, centre C, with equation $x^2 + y^2 - 2x - 4y - 5 = 0$ at the point $A(2, 5)$. Give your answer in the form $ax + by + c = 0$, where a, b and c are integers.

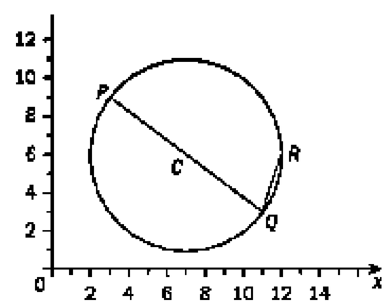
Example 5

Determine the shortest distance from the point $P(1, 1)$ to the circle, centre C, with equation $x^2 + y^2 + 6x - 8y + 21 = 0$.

Example 6

The diagram shows a circle, centre C, with equation $(x - 7)^2 + (y - 6)^2 = 25$. PQ is a diameter, where P is $(3, 9)$ and Q is $(11, 3)$. $R(12, 6)$ lies on the circle.

Show that the line perpendicular to QR and passing through R goes through P.



Example 7

AB is the diameter of a circle, where A is $(2, 6)$ and B is $(8, 2)$.

The tangent to the circle at B meets the x-axis at P and the y-axis at Q. Find the coordinates of P and Q.

Example 8

The points P, Q and R have coordinates $(0, 2)$, $(4, 0)$ and $(8, 8)$ respectively.

Use Pythagoras' theorem to show that PQR is a right-angled triangle and hence find the equation of the circle that passes through P, Q and R.

Homework: Equations of Circles Practice

1

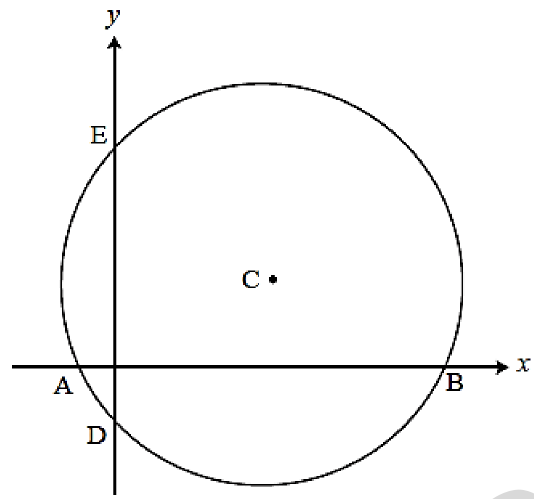


Fig. 11

Fig. 11 shows a sketch of the circle with equation $(x-10)^2+(y-2)^2=125$ and centre C. The points A, B, D and E are the intersections of the circle with the axes.

(i) Write down the radius of the circle and the coordinates of C. **[2]**

(ii) Verify that B is the point (21, 0) and find the coordinates of A, D and E. **[4]**

(iii) Find the equation of the perpendicular bisector of BE and verify that this line passes through C. **[6]**

Answer: i) (10,2) ii) (-1,0) , (0,-3), (0,7) iii) y=3x-28 Circles 1 Q1

2 Fig. 10 shows a sketch of a circle with centre C(4, 2). The circle intersects the x-axis at A(1, 0) and at B.

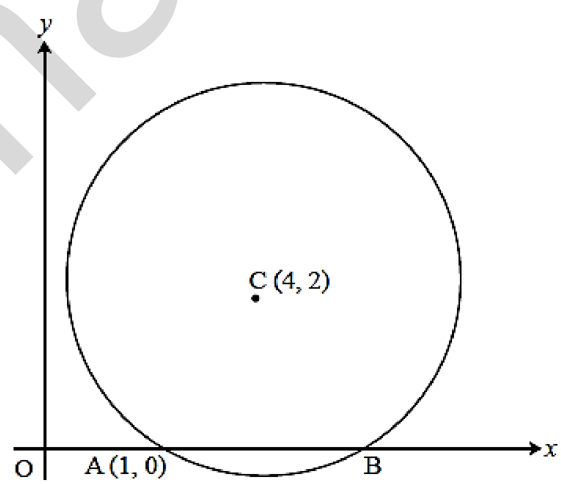


Fig. 10

(i) Write down the coordinates of B. [1]

(ii) Find the radius of the circle and hence write down the equation of the circle. [4]

(iii) AD is a diameter of the circle. Find the coordinates of D. [2]

(iv) Find the equation of the tangent to the circle at D. Give your answer in the form $y = ax + b$. [4]

Answer: i) (7,0) ii) $\sqrt{13}$ iii) (7,4) iv) y=-1.5x+14.5 — Circles 1 Q 2

3 The circle $(x-3)^2 + (y-2)^2 = 20$ has centre C.

(i) Write down the radius of the circle and the coordinates of C. [2]

(ii) Find the coordinates of the intersections of the circle with the x- and y-axes. [5]

(iii) Show that the points $A(1,6)$ and $B(7,4)$ lie on the circle. Find the coordinates of the midpoint of AB. Find also the distance of the chord AB from the centre of the circle. [5]

Answer: i) (3,2) ii) $(0, 2\pm\sqrt{11})$ or $\left(0, \frac{4\pm\sqrt{44}}{2}\right)$ iii) (4,5) and $\sqrt{10}$ — Circles 1 Q3

4 A circle has equation $(x-2)^2 + y^2 = 20$.

(i) Write down the radius of the circle and the coordinates of its centre. [2]

(ii) Find the points of intersection of the circle with the y-axis and sketch the circle. [3]

(iii) Show that, where the line $y = 2x + k$ intersects the circle,

$$5x^2 + (4k-4)x + k^2 - 16 = 0.$$ [3]

(iv) Hence find the values of k for which the line $y = 2x + k$ is a tangent to the circle. [4]

Answer: i)$\sqrt{20}$ and (2,0) ii) (0, $\pm$4) iv) 6 or -14 — Circles 1 Question 5

5 A circle has equation $(x-3)^2 + (y+2)^2 = 25$.

(i) State the coordinates of the centre of this circle and its radius. [2]

(ii) Verify that the point A with coordinates (6, –6) lies on this circle. Show also that the point B on the circle for which AB is a diameter has coordinates (0, 2). [3]

(iii) Find the equation of the tangent to the circle at A. [4]

(iv) A second circle touches the original circle at A. Its radius is 10 and its centre is at C, where BAC is a straight line. Find the coordinates of C and hence write down the equation of this second circle. [3]

Answer: i) (3,-2) and 5 iii) y=0.75x – 10.5 iv) (12,-14) and $(x-12)^2+(y+14)^2=100$ — Circles 2 Question 2

6 A circle has equation $(x-5)^2 + (y-2)^2 = 20$.

(i) State the coordinates of the centre and the radius of this circle. **[2]**

(ii) State, with a reason, whether or not this circle intersects the y-axis. **[2]**

(iii) Find the equation of the line parallel to the line $y = 2x$ that passes through the centre of the circle. **[2]**

(iv) Show that the line $y = 2x + 2$ is a tangent to the circle. State the coordinates of the point of contact. **[5]**

Answer: i) (5,2) and $\sqrt{20}$ ii) No, since $\sqrt{20} < 5$ iii) y=2x-8 iv) (1,4)

Circles 2
Question 4

7

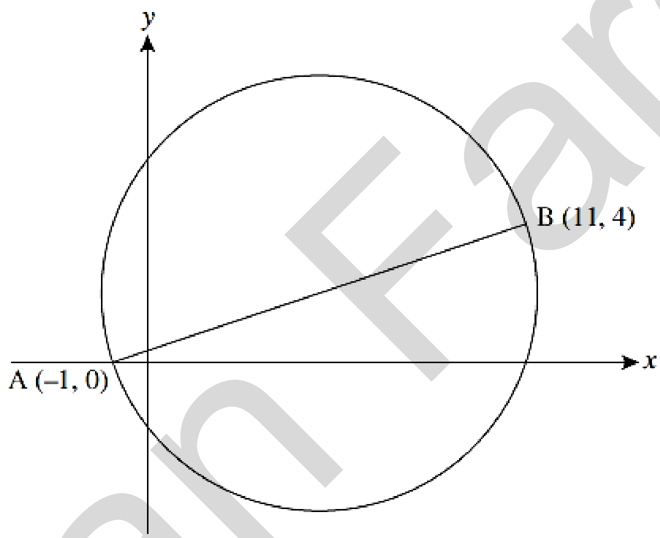


Fig. 11

Fig. 11 shows the points A and B, which have coordinates (–1, 0) and (11, 4) respectively.

(i) Show that the equation of the circle with AB as diameter may be written as

$$(x-5)^2 + (y-2)^2 = 40.$$ **[4]**

(ii) Find the coordinates of the points of intersection of this circle with the y-axis. Give your answer in the form $a \pm \sqrt{b}$. **[4]**

(iii) Find the equation of the tangent to the circle at B. Hence find the coordinates of the points of intersection of this tangent with the axes. **[6]**

Answer: ii) $2\pm\sqrt{15}$ iii) y= -3x+37 and (0,37) and ($\frac{37}{3}$,0)

Circles 3
Question 1

8 A circle has equation $x^2 + y^2 - 8x - 4y = 9$.

(i) Show that the centre of this circle is C (4, 2) and find the radius of the circle. [3]

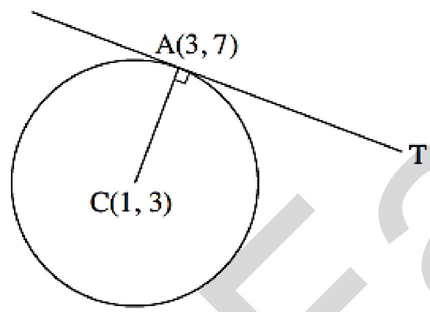
(ii) Show that the origin lies inside the circle. [2]

(iii) Show that AB is a diameter of the circle, where A has coordinates (2, 7) and B has coordinates (6, −3). [4]

(iv) Find the equation of the tangent to the circle at A. Give your answer in the form $y = mx + c$. [4]

Answer: i) $\sqrt{29}$ iv) $y = \frac{2}{5}x + \frac{31}{5}$ Circles 3 Q 2

9



Not to scale

Fig. 11

A circle has centre C(1, 3) and passes through the point A(3, 7) as shown in Fig. 11.

(i) Show that the equation of the tangent at A is $x + 2y = 17$. [4]

(ii) The line with equation $y = 2x - 9$ intersects this tangent at the point T.

Find the coordinates of T. [3]

(iii) The equation of the circle is $(x - 1)^2 + (y - 3)^2 = 20$.

Show that the line with equation $y = 2x - 9$ is a tangent to the circle. Give the coordinates of the point where this tangent touches the circle. [5]

Answer: ii) (7,5) iii) (5,1) Circles 3 Q 3

10 A(9, 8), B(5, 0) an C(3, 1) are three points.

(i) Show that AB and BC are perpendicular. [3]

(ii) Find the equation of the circle with AC as diameter. You need not simplify your answer.

Show that B lies on this circle. [6]

(iii) BD is a diameter of the circle. Find the coordinates of D. [3]

Answer: ii) $(x - 6)^2 + (y - 4.5)^2 = \frac{85}{4}$ iii) (7,9) Circles 4 Q2

11 A circle has equation $x^2 + y^2 = 45$.

(i) State the centre and radius of this circle. [2]

(ii) The circle intersects the line with equation $x + y = 3$ at two points, A and B. Find algebraically the coordinates of A and B.

Show that the distance AB is $\sqrt{162}$. [8]

Answer: i) (0,0) and $\sqrt{45}$ ii) (-3,6) and (6,-3) — Circles 4 Question 3

12 **(i)** Points A and B have coordinates (–2, 1) and (3, 4) respectively. Find the equation of the perpendicular bisector of AB and show that it may be written as $5x + 3y = 10$. **[6]**

(ii) Points C and D have coordinates (–5, 4) and (3, 6) respectively. The line through C and D has equation $4y = x + 21$. The point E is the intersection of CD and the perpendicular bisector of AB. Find the coordinates of point E. **[3]**

(iii) Find the equation of the circle with centre E which passes through A and B. Show also that CD is a diameter of this circle. **[5]**

Answer: ii) (-1,5) iii) $(x + 1)^2 + (y - 5)^2 = 17$ — Circles 4 Q5

13 The points A (–1, 6), B (1, 0) and C (13, 4) are joined by straight lines.

(i) Prove that the lines AB and BC are perpendicular. **[3]**

(ii) Find the area of triangle ABC. **[3]**

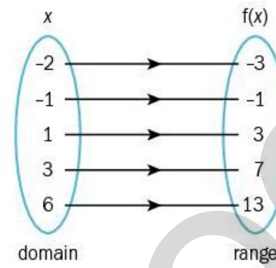
(iii) A circle passes through the points A, B and C. Justify the statement that AC is a diameter of this circle. Find the equation of this circle. **[6]**

(iv) Find the coordinates of the point on this circle that is furthest from B. **[1]**

Answer: ii) 40 iii) $(x - 6)^2 + (y - 5)^2 = 50$ iv) (11,10) — Circles 4 Q6

Functions

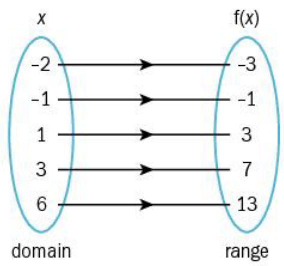
A **function** is defined as a mapping where every element of the domain (x-values) is mapped onto exactly one element of the range (y-values). The diagram on the right shows an example of a function.



We can write this function in two different ways:

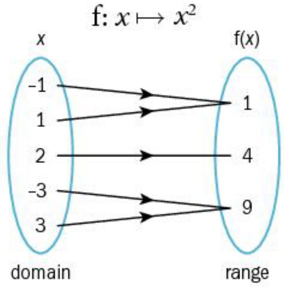
$f(x) = 2x + 1$ or $f: x \mapsto 2x + 1$

$f: x \mapsto 2x + 1$



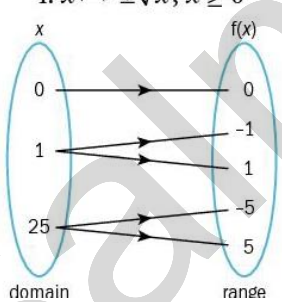
If every element in the domain is mapped onto exactly one element in the range and every element in the range is mapped onto exactly one element in the domain, we say the function is a **one-to-one** function.

$f: x \mapsto x^2$



If every element in the domain is mapped onto exactly one element in the range, but some elements in the range arise from more than one element in the domain, we say the function is a **many-to-one** function.

$f: x \mapsto \pm\sqrt{x},\ x \geq 0$



This is **not** a function, because it is one-to-many. Some elements in the domain are mapped onto more than one element in the range.

Example 1

$f(x) = 5x + 2x^3$

Find

a) $f(1)$ **b)** $f(-2)$ **c)** $f(-10)$

d) $f(8)$ **e)** $f(-3)$ **f)** $f(-1)$.

Example 2

A function f *is given by* $f : x \longmapsto x^2 - x + 1$. *Find*
(a) $f(2)$, *(b)* $f(-3)$, *(c) the image of* -2, *(d)* $f(r)$, *(e)* $f(\frac{x}{2})$.

Example 3

The function h *is given by* $h(x) = \frac{x+1}{x-1}$, $x \neq 1$. *Find*
(a) $h(2)$, *(b)* $h(\frac{1}{2})$, *(c)* $h(x + 1)$

Example 4

$F(x) = x^2 + x - 1$. *If* $F(x) = 5$, *find the values of* x.

Composite Functions

Example 5

The functions f and g are defined by

$f: x \mapsto 3x + 2, x \in \mathbb{R}$ $g: x \mapsto 7 - x, x \in \mathbb{R}$

a) Find $fg(x)$. **b)** Find $ff(x)$. **c)** Solve the equation $gf(x) = 2x$.

Example 6

The functions f, g, and h are defined by

$f: x \mapsto 4x - 1, x \in \mathbb{R}$ $g: x \mapsto \frac{1}{x+2}, x \neq -2$ $h: x \mapsto (2 - x)^2, x \in \mathbb{R}$

Find **a)** $fg(x)$ **b)** $hh(x)$

Example 7

The functions f, g and h are defined by:

$f(x) = 2x$ for $x \in \mathbb{R}$, $g(x) = x^2$ for $x \in \mathbb{R}$, $h(x) = \frac{1}{x}$ for $x \in \mathbb{R}$, $x \neq 0$.

Find the following.

(i) $fg(x)$ (ii) $gf(x)$ (iii) $gh(x)$

(iv) $f^2(x)$ (v) $fgh(x)$

Example 8

The functions f and g are defined by

f: $x \mapsto ax^2, x \in \mathbb{R}$ g: $x \mapsto 3x + b, x \in \mathbb{R}$ where a and b are constants.

Given that gf(1) = 5 and gg(2) = 14 find

a) the values of a and b **b)** the value of fg(−3).

Example 9

Example 2.3 *Find f^2, g^2, fg and gf for the following functions:*

(i) $f(x) = 3x - 1,\ g(x) = x - 3,$

(ii) $f(x) = 2x + 1,\ g(x) = x^2 + 1,$

(iii) $f(x) = 3x + 2,\ g(x) = \dfrac{x-1}{x+1},\ x \neq -1,$

(iv) $f(x) = 4x + 5,\ g(x) = \dfrac{x-5}{4}.$

Example 10

Find the range of these functions.

a) $f(x) = x + 5, x \geq 5$ **b)** $g(x) = x^3, -2 \leq x < 4$

Example 11

What values of x must be excluded from the domain of the function

$$f(x) = \frac{x+1}{x^2+x-2} \; ?$$

Example 12

a) Sketch the graph of the function defined by

$$f(x) = \begin{cases} 4 - x & \text{when } -2 \leq x \leq 1 \\ 2x + 1 & \text{when } 1 < x \leq 3 \end{cases}$$

b) Find the range.

Example 13

$h(x) = 2x^2 - 12x + 22, x \in \mathbb{R}$

a) Express h(x) in the form $a(x + b)^2 + c$. **b)** Find the range.

Example 14

$f(x) = \frac{5}{(x-2)^2}, x \in \mathbb{R}, f(x) \geq 5$

Find the greatest possible domain.

Example 15

The functions f and g are defined for $x \in \mathbb{R}$ by

$f: x \mapsto 4x - 2x^2;$

$g: x \mapsto 5x + 3.$

(i) Find the range of f.

(ii) Find the value of the constant k for which the equation $gf(x) = k$ has equal roots.

[Cambridge AS & A Level Mathematics 9709, Paper 12 Q3 June 2010]

Inverse of a function

The **inverse function** provides structured evidence that leads to a given result of f. The inverse function is written as $f^{-1}(x)$ and maps the range {2, 5, 10, 17, 26} back onto the domain {1, 2, 3, 4, 5}.

Example 16

Find $f^{-1}(x)$ when $f(x) = 2x + 1$, $x \in \mathbb{R}$.

Example 17

Find $f^{-1}(x)$ when $f(x) = 2x - 3$ and the domain of f is $x \geqslant 4$.

Example 18

Functions f and g are defined by

$$f : x \mapsto k - x \quad \text{for } x \in \mathbb{R}, \text{ where } k \text{ is a constant,}$$

$$g : x \mapsto \frac{9}{x+2} \quad \text{for } x \in \mathbb{R}, x \neq -2.$$

(i) Find the values of k for which the equation $f(x) = g(x)$ has two equal roots and solve the equation $f(x) = g(x)$ in these cases.

(ii) Solve the equation $fg(x) = 5$ when $k = 6$.

(iii) Express $g^{-1}(x)$ in terms of x.

[Cambridge AS & A Level Mathematics 9709, Paper 1 Q11 June 2006]

Example 19

Functions f and g are defined by

$f: x \mapsto 3x - 4, x \in \mathbb{R}$ $\qquad$ $g: x \mapsto \dfrac{1}{x+1}, x \in \mathbb{R}, x \neq -1$

a) Express in terms of x **i)** $f^{-1}(x)$ **ii)** $g^{-1}(x)$

b) Sketch in a single diagram the graphs of $y = f(x)$ and $y = f^{-1}(x)$, making clear the relationship between the graphs.

Example 20

The functions f is defined by f: $x \mapsto x^2 - 6x + 2$ for $0 \le x \le 3$.

a) Express f(x) in the form $(x + a)^2 + b$, where a and b are constants.

b) State the range of f.

c) State the domain of $f^{-1}(x)$.

d) Find an expression for $f^{-1}(x)$.

e) Express $ff^{-1}(x)$ in terms of x.

f) Sketch on the same diagram the graphs of $y = f(x)$ and $y = f^{-1}(x)$, making clear the relationship between the graphs.

Example 21

The function f is defined by f: $x \mapsto 2x^2 - 8x + 11$ for $x \in \mathbb{R}$.

(i) Express f(x) in the form $a(x + b)^2 + c$, where a, b and c are constants.

(ii) State the range of f.

(iii) Explain why f does not have an inverse.

The function g is defined by g: $x \mapsto 2x^2 - 8x + 11$ for $x \leqslant A$, where A is a constant.

(iv) State the largest value of A for which g has an inverse.

(v) When A has this value, obtain an expression, in terms of x, for $g^{-1}(x)$ and state the range of g^{-1}

[Cambridge AS & A Level Mathematics 9709, Paper 1 Q11 November 2007]

Example 22

The function f is defined by f: $x \mapsto 3x - 2$ for $x \in \mathbb{R}$.

(i) Sketch, in a single diagram, the graphs of $y = f(x)$ and $y = f^{-1}(x)$, making clear the relationship between the two graphs.

The function g is defined by g: $x \mapsto 6x - x^2$ for $x \in \mathbb{R}$.

(ii) Express gf(x) in terms of x, and hence show that the maximum value of gf(x) is 9.

The function h is defined by h: $x \mapsto 6x - x^2$ for $x \geqslant 3$.

(iii) Express $6x - x^2$ in the form $a - (x - b)^2$, where a and b are positive constants.

(iv) Express $h^{-1}(x)$ in terms of x.

[Cambridge AS & A Level Mathematics 9709, Paper 1 Q10 November 2008]

Range and Domain of a composite function

Example 23

June 2016/13 Question 10

The function f is such that $f(x) = 2x + 3$ for $x \geqslant 0$. The function g is such that $g(x) = ax^2 + b$ for $x \leqslant q$, where a, b and q are constants. The function fg is such that $fg(x) = 6x^2 - 21$ for $x \leqslant q$.

(i) Find the values of a and b. [3]

(ii) Find the greatest possible value of q. [2]

It is now given that $q = -3$.

(iii) Find the range of fg. [1]

(iv) Find an expression for $(fg)^{-1}(x)$ and state the domain of $(fg)^{-1}$. [3]

Example 24

November 2015/13 Question 8

The function f is defined by $f(x) = 3x + 1$ for $x \leqslant a$, where a is a constant. The function g is defined by $g(x) = -1 - x^2$ for $x \leqslant -1$.

(i) Find the largest value of a for which the composite function gf can be formed. [2]

For the case where $a = -1$,

(ii) solve the equation $fg(x) + 14 = 0$, [3]

(iii) find the set of values of x which satisfy the inequality $gf(x) \leqslant -50$. [4]

Example 25

June 2019/13 Q4

The function f is defined by $f(x) = \dfrac{48}{x-1}$ for $3 \leqslant x \leqslant 7$. The function g is defined by $g(x) = 2x - 4$ for $a \leqslant x \leqslant b$, where a and b are constants.

(i) Find the greatest value of a and the least value of b which will permit the formation of the composite function gf. [2]

It is now given that the conditions for the formation of gf are satisfied.

(ii) Find an expression for $gf(x)$. [1]

(iii) Find an expression for $(gf)^{-1}(x)$. [2]

Example 26

November 2016/13 Question 8

(i) Express $4x^2 + 12x + 10$ in the form $(ax+b)^2 + c$, where a, b and c are constants. [3]

(ii) Functions f and g are both defined for $x > 0$. It is given that $f(x) = x^2 + 1$ and $fg(x) = 4x^2 + 12x + 10$. Find $g(x)$. [1]

(iii) Find $(fg)^{-1}(x)$ and give the domain of $(fg)^{-1}$. [4]

Extra questions

Example 27

2020 Specimen Paper 1 Question 11

The function f is defined, for $x \in \mathbb{R}$, by $f : x \mapsto x^2 + ax + b$, where a and b are constants.

(a) It is given that $a = 6$ and $b = -8$.

Find the range of f. [3]

(b) It is given instead that $a = 5$ and that the roots of the equation $f(x) = 0$ are k and $-2k$, where k is a constant.

Find the values of b and k. [3]

(c) Show that if the equation $f(x + a) = a$ has no real roots then $a^2 < 4(b - a)$. [3]

Example 29

November 2016/11 Question 8

The functions f and g are defined by

$$f(x) = \frac{4}{x} - 2 \quad \text{for } x > 0,$$

$$g(x) = \frac{4}{5x+2} \quad \text{for } x \geqslant 0.$$

(i) Find and simplify an expression for $fg(x)$ and state the range of fg. [3]

(ii) Find an expression for $g^{-1}(x)$ and find the domain of g^{-1}. [5]

Example 30

June 2016/12 Question 1

Functions f and g are defined by

$$f : x \mapsto 10 - 3x, \quad x \in \mathbb{R},$$
$$g : x \mapsto \frac{10}{3 - 2x}, \quad x \in \mathbb{R},\ x \neq \tfrac{3}{2}.$$

Solve the equation $ff(x) = gf(2)$. [3]

Example 31

November 2016/12 Question 10

A function f is defined by $f : x \mapsto 5 - 2\sin 2x$ for $0 \leqslant x \leqslant \pi$.

(i) Find the range of f. [2]

(ii) Sketch the graph of $y = f(x)$. [2]

(iii) Solve the equation $f(x) = 6$, giving answers in terms of π. [3]

The function g is defined by $g : x \mapsto 5 - 2\sin 2x$ for $0 \leqslant x \leqslant k$, where k is a constant.

(iv) State the largest value of k for which g has an inverse. [1]

(v) For this value of k, find an expression for $g^{-1}(x)$. [3]

Example 32

June 2016/11 Question 11

The function f is defined by $f : x \mapsto 4\sin x - 1$ for $-\frac{1}{2}\pi \leqslant x \leqslant \frac{1}{2}\pi$.

(i) State the range of f. [2]

(ii) Find the coordinates of the points at which the curve $y = f(x)$ intersects the coordinate axes. [3]

(iii) Sketch the graph of $y = f(x)$. [2]

(iv) Obtain an expression for $f^{-1}(x)$, stating both the domain and range of f^{-1}. [4]

Example 33

June 2016/12 Question 11

The function f is defined by $\mathrm{f} : x \mapsto 6x - x^2 - 5$ for $x \in \mathbb{R}$.

(i) Find the set of values of x for which $\mathrm{f}(x) \leqslant 3$. [3]

(ii) Given that the line $y = mx + c$ is a tangent to the curve $y = \mathrm{f}(x)$, show that $4c = m^2 - 12m + 16$. [3]

The function g is defined by $\mathrm{g} : x \mapsto 6x - x^2 - 5$ for $x \geqslant k$, where k is a constant.

(iii) Express $6x - x^2 - 5$ in the form $a - (x - b)^2$, where a and b are constants. [2]

(iv) State the smallest value of k for which g has an inverse. [1]

(v) For this value of k, find an expression for $\mathrm{g}^{-1}(x)$. [2]

Example 34

June 2016/12 Question 8

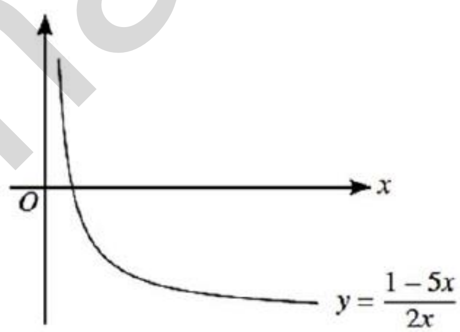
The function f is such that $\mathrm{f}(x) = a^2x^2 - ax + 3b$ for $x \leqslant \frac{1}{2a}$, where a and b are constants.

(i) For the case where $\mathrm{f}(-2) = 4a^2 - b + 8$ and $\mathrm{f}(-3) = 7a^2 - b + 14$, find the possible values of a and b. [5]

(ii) For the case where $a = 1$ and $b = -1$, find an expression for $\mathrm{f}^{-1}(x)$ and give the domain of f^{-1}. [5]

Example 35

June 2015/13 Question 6



The diagram shows the graph of $y = \mathrm{f}^{-1}(x)$, where f^{-1} is defined by $\mathrm{f}^{-1}(x) = \frac{1-5x}{2x}$ for $0 < x \leqslant 2$.

(i) Find an expression for $\mathrm{f}(x)$ and state the domain of f. [5]

(ii) The function g is defined by $\mathrm{g}(x) = \frac{1}{x}$ for $x \geqslant 1$. Find an expression for $\mathrm{f}^{-1}\mathrm{g}(x)$, giving your answer in the form $ax + b$, where a and b are constants to be found. [2]

Homework: Functions Variant 12

1 The function f is defined by $f : x \mapsto ax + b$, for $x \in \mathbb{R}$, where a and b are constants. It is given that $f(2) = 1$ and $f(5) = 7$.

(i) Find the values of a and b. [2]

(ii) Solve the equation $ff(x) = 0$. [3]

Answers: (i) $a = 2$, $b = -3$; (ii) 2.25 . J03/Q5

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2 The equation of a curve is $y = 8x - x^2$.

(i) Express $8x - x^2$ in the form $a - (x + b)^2$, stating the numerical values of a and b. [3]

(ii) Hence, or otherwise, find the coordinates of the stationary point of the curve. [2]

(iii) Find the set of values of x for which $y \geqslant -20$. [3]

The function g is defined by $g : x \mapsto 8x - x^2$, for $x \geqslant 4$.

(iv) State the domain and range of g^{-1}. [2]

(v) Find an expression, in terms of x, for $g^{-1}(x)$. [3]

Answers: (i) $16 - (x - 4)^2$, $a = 16$, $b = -4$; (ii) (4, 16); (iii) $-2 \leqslant x \leqslant 10$; J03/Q11
(iv) Domain $x \leqslant 16$, range $g^{-1}(x) \geqslant 4$; (v) $g^{-1}(x) = 4 + \sqrt{16 - x}$.

---

3 The functions f and g are defined as follows:

$$f : x \mapsto x^2 - 2x, \quad x \in \mathbb{R},$$
$$g : x \mapsto 2x + 3, \quad x \in \mathbb{R}.$$

(i) Find the set of values of x for which $f(x) > 15$. [3]

(ii) Find the range of f and state, with a reason, whether f has an inverse. [4]

(iii) Show that the equation $gf(x) = 0$ has no real solutions. [3]

(iv) Sketch, in a single diagram, the graphs of $y = g(x)$ and $y = g^{-1}(x)$, making clear the relationship between the graphs. [2]

Answers: (i) $x < -3$ and $x > 5$; (ii) $f(x) \geqslant -1$, f does not have an inverse. J04/Q10

4 A function f is defined by $f : x \mapsto 3 - 2\sin x$, for $0^\circ \leqslant x \leqslant 360^\circ$.

(i) Find the range of f. [2]

(ii) Sketch the graph of $y = f(x)$. [2]

A function g is defined by $g : x \mapsto 3 - 2\sin x$, for $0^\circ \leqslant x \leqslant A^\circ$, where A is a constant.

(iii) State the largest value of A for which g has an inverse. [1]

(iv) When A has this value, obtain an expression, in terms of x, for $g^{-1}(x)$. [2]

Answers: (i) $1 \leqslant f(x) \leqslant 5$; (iii) 90; (iv) $\sin^{-1}\left(\frac{3-x}{2}\right)$. **J05/Q7**

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5 Functions f and g are defined by

$$f : x \mapsto k - x \quad \text{for } x \in \mathbb{R}, \text{ where } k \text{ is a constant,}$$
$$g : x \mapsto \frac{9}{x+2} \quad \text{for } x \in \mathbb{R},\ x \neq -2.$$

(i) Find the values of k for which the equation $f(x) = g(x)$ has two equal roots and solve the equation $f(x) = g(x)$ in these cases. [6]

(ii) Solve the equation $fg(x) = 5$ when $k = 6$. [3]

(iii) Express $g^{-1}(x)$ in terms of x. [2]

Answers: (i) $k = 4$ or -8, $x = 1$ or -5; (ii) 7; (iii) $\frac{9-2x}{x}$. **J06/Q11**

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6 The function f is defined by $f(x) = a + b\cos 2x$, for $0 \leqslant x \leqslant \pi$. It is given that $f(0) = -1$ and $f(\frac{1}{2}\pi) = 7$.

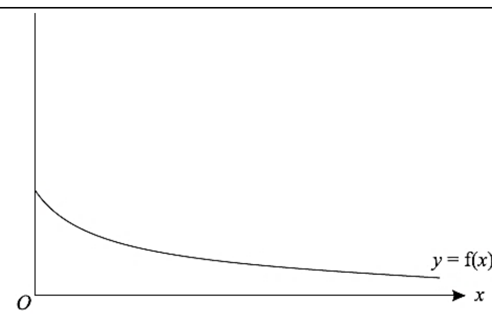
(i) Find the values of a and b. [3]

(ii) Find the x-coordinates of the points where the curve $y = f(x)$ intersects the x-axis. [3]

(iii) Sketch the graph of $y = f(x)$. [2]

Answers: (i) $a = 3$, $b = -4$; (ii) $x = 0.361$, 2.78. **J07/Q8**

7



The diagram shows the graph of $y = \mathrm{f}(x)$, where $\mathrm{f} : x \mapsto \dfrac{6}{2x+3}$ for $x \geqslant 0$.

(i) Find an expression, in terms of x, for $\mathrm{f}'(x)$ and explain how your answer shows that f is a decreasing function. [3]

(ii) Find an expression, in terms of x, for $\mathrm{f}^{-1}(x)$ and find the domain of f^{-1}. [4]

(iii) Copy the diagram and, on your copy, sketch the graph of $y = \mathrm{f}^{-1}(x)$, making clear the relationship between the graphs. [2]

The function g is defined by $\mathrm{g} : x \mapsto \frac{1}{2}x$ for $x \geqslant 0$.

(iv) Solve the equation $\mathrm{fg}(x) = \frac{3}{2}$. [3]

Answers: (i) $\dfrac{-12}{(2x+3)^2}$; (ii) $\dfrac{3}{x} - \dfrac{3}{2}$, $0 < x \leqslant 2$; (iv) $x = 1$. **J07/Q11**

8 Functions f and g are defined by

$$\mathrm{f} : x \mapsto 4x - 2k \quad \text{for } x \in \mathbb{R}, \text{ where } k \text{ is a constant,}$$

$$\mathrm{g} : x \mapsto \frac{9}{2-x} \quad \text{for } x \in \mathbb{R}, x \neq 2.$$

(i) Find the values of k for which the equation $\mathrm{fg}(x) = x$ has two equal roots. [4]

(ii) Determine the roots of the equation $\mathrm{fg}(x) = x$ for the values of k found in part **(i)**. [3]

Answers: (i) 5 or –7; (ii) $x = -4$ or 8. **J08/Q8**

9 The function f is defined by $\mathrm{f} : x \mapsto 2x^2 - 12x + 13$ for $0 \leqslant x \leqslant A$, where A is a constant.

(i) Express $\mathrm{f}(x)$ in the form $a(x+b)^2 + c$, where a, b and c are constants. [3]

(ii) State the value of A for which the graph of $y = \mathrm{f}(x)$ has a line of symmetry. [1]

(iii) When A has this value, find the range of f. [2]

The function g is defined by $\mathrm{g} : x \mapsto 2x^2 - 12x + 13$ for $x \geqslant 4$.

(iv) Explain why g has an inverse. [1]

(v) Obtain an expression, in terms of x, for $\mathrm{g}^{-1}(x)$. [3]

Answers: (i) $2(x-3)^2 - 5$; (ii) $A = 6$; (iii) $-5 \leq x \leq 13$; (iv) Proof; (v) $\sqrt{\dfrac{x+5}{2}} + 3$ **J09/Q10**

10 The functions f and g are defined for $x \in \mathbb{R}$ by

$$f : x \mapsto 4x - 2x^2,$$
$$g : x \mapsto 5x + 3.$$

(i) Find the range of f. [2]

(ii) Find the value of the constant k for which the equation $gf(x) = k$ has equal roots. [3]

Answers: (i) $f(x) < 2$; (ii) 13. **J10/12/Q3**

11 The function $f : x \mapsto 4 - 3\sin x$ is defined for the domain $0 \leqslant x \leqslant 2\pi$.

(i) Solve the equation $f(x) = 2$. [3]

(ii) Sketch the graph of $y = f(x)$. [2]

(iii) Find the set of values of k for which the equation $f(x) = k$ has no solution. [2]

The function $g : x \mapsto 4 - 3\sin x$ is defined for the domain $\frac{1}{2}\pi \leqslant x \leqslant A$.

(iv) State the largest value of A for which g has an inverse. [1]

(v) For this value of A, find the value of $g^{-1}(3)$. [2]

J10/12/Q11

Answers: (i) 0.730, 2.41; (iii) $k < 1$, $k > 7$; (iv) $\frac{3\pi}{2}$; (v) 2.80.

12 **(i)** Express $2x^2 + 8x - 10$ in the form $a(x + b)^2 + c$. [3]

(ii) For the curve $y = 2x^2 + 8x - 10$, state the least value of y and the corresponding value of x. [2]

(iii) Find the set of values of x for which $y \geqslant 14$. [3]

Given that $f : x \mapsto 2x^2 + 8x - 10$ for the domain $x \geqslant k$,

(iv) find the least value of k for which f is one-one, [1]

(v) express $f^{-1}(x)$ in terms of x in this case. [3]

Answers: (i) $a = 2, b = 2, c = -18$; (ii) $x = -2$, $y = -18$; (iii) $x \geqslant 2, x \leqslant -6$; (iv) -2; (v) $f^{-1}(x) = \sqrt{\frac{x+18}{2}} - 2$. **N02/Q11**

13 Functions f and g are defined by

$$f : x \mapsto 2x - 5, \quad x \in \mathbb{R},$$
$$g : x \mapsto \frac{4}{2-x}, \quad x \in \mathbb{R}, \ x \neq 2.$$

(i) Find the value of x for which $fg(x) = 7$. [3]

(ii) Express each of $f^{-1}(x)$ and $g^{-1}(x)$ in terms of x. [3]

(iii) Show that the equation $f^{-1}(x) = g^{-1}(x)$ has no real roots. [3]

(iv) Sketch, on a single diagram, the graphs of $y = f(x)$ and $y = f^{-1}(x)$, making clear the relationship between these two graphs. [3]

Answers: (i) $1\frac{1}{3}$; (ii) $f^{-1}(x) = \frac{1}{2}(x+5)$, $g^{-1}(x) = \frac{2x-4}{x}$; (iv) Sketch - symmetry about $y = x$. **N03/Q10**

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14 The function $f : x \mapsto 5\sin^2 x + 3\cos^2 x$ is defined for the domain $0 \leqslant x \leqslant \pi$.

(i) Express $f(x)$ in the form $a + b\sin^2 x$, stating the values of a and b. [2]

(ii) Hence find the values of x for which $f(x) = 7\sin x$. [3]

(iii) State the range of f. [2]

Answers: (i) $a = 3$, $b = 2$; (ii) 0.524, 2.62; (iii) $3 \leqslant f \leqslant 5$. **N04/Q6**

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15 The function $f : x \mapsto 2x - a$, where a is a constant, is defined for all real x.

(i) In the case where $a = 3$, solve the equation $ff(x) = 11$. [3]

The function $g : x \mapsto x^2 - 6x$ is defined for all real x.

(ii) Find the value of a for which the equation $f(x) = g(x)$ has exactly one real solution. [3]

The function $h : x \mapsto x^2 - 6x$ is defined for the domain $x \geqslant 3$.

(iii) Express $x^2 - 6x$ in the form $(x - p)^2 - q$, where p and q are constants. [2]

(iv) Find an expression for $h^{-1}(x)$ and state the domain of h^{-1}. [4]

Answers: (i) $x = 5$; (ii) $a = 16$; (iii) $p = 3$ and $q = 9$; (iv) $h^{-1}(x) = \sqrt{(x+9)} + 3$, $x \geqslant -9$. **N04/Q9**

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16 A function f is defined by $f : x \mapsto (2x - 3)^3 - 8$, for $2 \leqslant x \leqslant 4$.

(i) Find an expression, in terms of x, for $f'(x)$ and show that f is an increasing function. [4]

(ii) Find an expression, in terms of x, for $f^{-1}(x)$ and find the domain of f^{-1}. [4]

Answers: (i) $6(2x-3)^2$; (ii) $\frac{\sqrt[3]{(x+8}+3}{2}$, $-7 \leqslant x \leqslant 117$. N05/Q8

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17 The function f is defined by $f : x \mapsto x^2 - 3x$ for $x \in \mathbb{R}$.

(i) Find the set of values of x for which $f(x) > 4$. [3]

(ii) Express $f(x)$ in the form $(x - a)^2 - b$, stating the values of a and b. [2]

(iii) Write down the range of f. [1]

(iv) State, with a reason, whether f has an inverse. [1]

The function g is defined by $g : x \mapsto x - 3\sqrt{x}$ for $x \geqslant 0$.

(v) Solve the equation $g(x) = 10$. [3]

Answer: (i) $x < -1$ and $x > 4$; (ii) $a = 1\frac{1}{2}$, $b = 2\frac{1}{4}$; (iii) f(x) à $-2\frac{1}{4}$; (iv) no inverse, f not one-one; (v) $x = 25$. N06/Q10

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18 The function f is defined by $f : x \mapsto 2x^2 - 8x + 11$ for $x \in \mathbb{R}$.

(i) Express $f(x)$ in the form $a(x + b)^2 + c$, where a, b and c are constants. [3]

(ii) State the range of f. [1]

(iii) Explain why f does not have an inverse. [1]

The function g is defined by $g : x \mapsto 2x^2 - 8x + 11$ for $x \leqslant A$, where A is a constant.

(iv) State the largest value of A for which g has an inverse. [1]

(v) When A has this value, obtain an expression, in terms of x, for $g^{-1}(x)$ and state the range of g^{-1}. [4]

Answers: (i) $2(x-2)^2+3$; (ii) f(x) › 3; (iii) f is not one-one; (iv) 2; (v) $2-\sqrt{\left(\frac{x-3}{2}\right)}$, $g^{-1}(x)$ ø 2. N07/Q11

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19 The function f is such that $f(x) = a - b\cos x$ for $0° \leqslant x \leqslant 360°$, where a and b are positive constants. The maximum value of $f(x)$ is 10 and the minimum value is -2.

(i) Find the values of a and b. [3]

(ii) Solve the equation $f(x) = 0$. [3]

(iii) Sketch the graph of $y = f(x)$. [2]

Answers: (i) 4, 6; (ii) 48.2°, 311.8°. N08/Q5

20 The function f is defined by

$$f : x \mapsto 3x - 2 \text{ for } x \in \mathbb{R}.$$

(i) Sketch, in a single diagram, the graphs of $y = f(x)$ and $y = f^{-1}(x)$, making clear the relationship between the two graphs. [2]

The function g is defined by

$$g : x \mapsto 6x - x^2 \text{ for } x \in \mathbb{R}.$$

(ii) Express $gf(x)$ in terms of x, and hence show that the maximum value of $gf(x)$ is 9. [5]

The function h is defined by

$$h : x \mapsto 6x - x^2 \text{ for } x \geqslant 3.$$

(iii) Express $6x - x^2$ in the form $a - (x - b)^2$, where a and b are positive constants. [2]

(iv) Express $h^{-1}(x)$ in terms of x. [3]

N08/10

Answers: (ii) $-9x^2 + 30x - 16$; (iii) $9 - (x-3)^2$; (iv) $3 + \sqrt{9-x}$.

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21 The function f is defined by $f : x \mapsto 5 - 3\sin 2x$ for $0 \leqslant x \leqslant \pi$.

(i) Find the range of f. [2]

(ii) Sketch the graph of $y = f(x)$. [3]

(iii) State, with a reason, whether f has an inverse. [1]

Answers: (i) $2 \leqslant f(x) \leqslant 8$; (iii) No inverse, not one-one. N09/12/Q4

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22 The function f is such that $f(x) = \dfrac{3}{2x+5}$ for $x \in \mathbb{R}$, $x \neq -2.5$.

(i) Obtain an expression for $f'(x)$ and explain why f is a decreasing function. [3]

(ii) Obtain an expression for $f^{-1}(x)$. [2]

(iii) A curve has the equation $y = f(x)$. Find the volume obtained when the region bounded by the curve, the coordinate axes and the line $x = 2$ is rotated through 360° about the x-axis. [4]

Answers: (i) $\dfrac{-6}{(2x+5)^2}$; (ii) $\dfrac{1}{2}\left(\dfrac{3}{x} - 5\right)$; (iii) 1.26. N09/12/Q8

23 The function f is defined by

$$f(x) = x^2 - 4x + 7 \text{ for } x > 2.$$

(i) Express $f(x)$ in the form $(x - a)^2 + b$ and hence state the range of f. [3]

(ii) Obtain an expression for $f^{-1}(x)$ and state the domain of f^{-1}. [3]

The function g is defined by

$$g(x) = x - 2 \text{ for } x > 2.$$

The function h is such that f = hg and the domain of h is $x > 0$.

(iii) Obtain an expression for $h(x)$. [1]

Answers: **(i)** $(x-2)^2+3$, $f(x) > 3$; **(ii)** $2+\sqrt{x-3}$, $x > 3$; **(iii)** $x^2 + 3$. **N10/12/Q7**

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24 The function f is such that $f(x) = (3x + 2)^3 - 5$ for $x \geqslant 0$.

(i) Obtain an expression for $f'(x)$ and hence explain why f is an increasing function. [3]

(ii) Obtain an expression for $f^{-1}(x)$ and state the domain of f^{-1}. [4]

Answers: **(i)** $9(3x + 2)^2$; **(ii)** $\dfrac{\sqrt[3]{x+5}-2}{3}$, $x > 3$. **J08/Q6**

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25 The function f is defined by $f : x \mapsto \dfrac{x+3}{2x-1}$, $x \in \mathbb{R}$, $x \neq \frac{1}{2}$.

(i) Show that $ff(x) = x$. [3]

(ii) Hence, or otherwise, obtain an expression for $f^{-1}(x)$. [2]

Answer: **(ii)** $f^{-1}(x) = \dfrac{x+3}{2x-1}$. **J11/12/Q6**

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26 The functions f and g are defined for $x \in \mathbb{R}$ by

$$f : x \mapsto 3x + a,$$
$$g : x \mapsto b - 2x,$$

where a and b are constants. Given that $ff(2) = 10$ and $g^{-1}(2) = 3$, find

(i) the values of a and b, [4]

(ii) an expression for $fg(x)$. [2]

Answers: **(i)** $a = -2$, $b = 8$; **(ii)** $22 - 6x$ **N11/12/Q2**

27 Functions f and g are defined by

$$f : x \mapsto 2x + 5 \quad \text{for } x \in \mathbb{R},$$
$$g : x \mapsto \frac{8}{x-3} \quad \text{for } x \in \mathbb{R},\ x \neq 3.$$

(i) Obtain expressions, in terms of x, for $f^{-1}(x)$ and $g^{-1}(x)$, stating the value of x for which $g^{-1}(x)$ is not defined. [4]

(ii) Sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$ on the same diagram, making clear the relationship between the two graphs. [3]

(iii) Given that the equation $fg(x) = 5 - kx$, where k is a constant, has no solutions, find the set of possible values of k. [5]

Answers: **(i)** $f^{-1}(x) = ½(x - 5)$, $g^{-1}(x) = \frac{8}{x} + 3$ $\quad x = 0$ **(ii)** Sketch **(iii)** $0 < k < \frac{64}{9}$ **J12/12/Q10**

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28 A function f is defined by $f(x) = \frac{5}{1 - 3x}$, for $x \geqslant 1$.

(i) Find an expression for $f'(x)$. [2]

(ii) Determine, with a reason, whether f is an increasing function, a decreasing function or neither. [1]

(iii) Find an expression for $f^{-1}(x)$, and state the domain and range of f^{-1}. [5]

Answers: **(i)** $\frac{15}{(1-3x)^2}$. **(ii)** Increasing. **(iii)** $f^{-1}(x) = \frac{x-5}{3x}$. Range $f^{-1}(x) \geq 1$. Domain $-2.5 \leq x \leq 0$. **J13/12/Q9**

---

29 Functions f and g are defined by

$$f : x \mapsto 2x - 3, \quad x \in \mathbb{R},$$
$$g : x \mapsto x^2 + 4x, \quad x \in \mathbb{R}.$$

(i) Solve the equation $ff(x) = 11$. [2]

(ii) Find the range of g. [2]

(iii) Find the set of values of x for which $g(x) > 12$. [3]

(iv) Find the value of the constant p for which the equation $gf(x) = p$ has two equal roots. [3]

Function h is defined by $h : x \mapsto x^2 + 4x$ for $x \geqslant k$, and it is given that h has an inverse.

(v) State the smallest possible value of k. [1]

(vi) Find an expression for $h^{-1}(x)$. [4]

Answers: **(i)** 5 **(ii)** Range ≥ -4 **(iii)** $x < -6, x > 2$ **(iv)** $p = -4$ **(v)** -2 **(vi)** $h^{-1}(x) = \sqrt{x+4} - 2$ **J14/12/Q10**

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30 The function f is defined by $f : x \mapsto 2x^2 - 6x + 5$ for $x \in \mathbb{R}$.

(i) Find the set of values of p for which the equation $f(x) = p$ has no real roots. [3]

The function g is defined by $g : x \mapsto 2x^2 - 6x + 5$ for $0 \leqslant x \leqslant 4$.

(ii) Express $g(x)$ in the form $a(x+b)^2 + c$, where a, b and c are constants. [3]

(iii) Find the range of g. [2]

The function h is defined by $h : x \mapsto 2x^2 - 6x + 5$ for $k \leqslant x \leqslant 4$, where k is a constant.

(iv) State the smallest value of k for which h has an inverse. [1]

(v) For this value of k, find an expression for $h^{-1}(x)$. [3]

Answer: **(i)** $p<1/2$; **(ii)** $2(x-\frac{3}{2})^2 + \frac{1}{2}$, **(iii)** $\frac{1}{2} \leq g(x) \leq 13$ **(iv)** $\frac{3}{2}$, (v) $\frac{3}{2} + \sqrt{\left(\frac{2x-1}{4}\right)}$ **J15/12/Q11**

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31 A function f is such that $f(x) = \sqrt{\left(\frac{x+3}{2}\right)} + 1$, for $x \geqslant -3$. Find

(i) $f^{-1}(x)$ in the form $ax^2 + bx + c$, where a, b and c are constants, [3]

(ii) the domain of f^{-1}. [1]

Answers: **(i)** $2x^2 - 4x - 1$; **(ii)** $x \geq 1$. **N12/12/Q2**

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32 A function f is defined by $f : x \mapsto 3\cos x - 2$ for $0 \leqslant x \leqslant 2\pi$.

(i) Solve the equation $f(x) = 0$. [3]

(ii) Find the range of f. [2]

(iii) Sketch the graph of $y = f(x)$. [2]

A function g is defined by $g : x \mapsto 3\cos x - 2$ for $0 \leqslant x \leqslant k$.

(iv) State the maximum value of k for which g has an inverse. [1]

(v) Obtain an expression for $g^{-1}(x)$. [2]

Answers: **(i)** 0.841, 5.44. **(ii)** $-5 \leq f(x) \leq 1$. **(iii)** Graph. **(iv)** π. **(v)** $g^{-1}(x) = \cos^{-1}\frac{x+2}{3}$. **N13/12/Q8**

33 A curve has equation $y = 2x^2 - 3x$.

(i) Find the set of values of x for which $y > 9$. [3]

(ii) Express $2x^2 - 3x$ in the form $a(x+b)^2 + c$, where a, b and c are constants, and state the coordinates of the vertex of the curve. [4]

The functions f and g are defined for all real values of x by

$$f(x) = 2x^2 - 3x \quad \text{and} \quad g(x) = 3x + k,$$

where k is a constant.

(iii) Find the value of k for which the equation $gf(x) = 0$ has equal roots. [3]

Answers: **(i)** $x < -1\frac{1}{2},\ x > 3$. **(ii)** $2\left(x-\frac{3}{4}\right)^2 - \frac{9}{8}$, $\left(\frac{3}{4}, -\frac{9}{8}\right)$. **(iii)** $k = \frac{27}{8}$. **N13/12/Q10**

34 The function $f : x \mapsto 6 - 4\cos(\frac{1}{2}x)$ is defined for $0 \leqslant x \leqslant 2\pi$.

(i) Find the exact value of x for which $f(x) = 4$. [3]

(ii) State the range of f. [2]

(iii) Sketch the graph of $y = f(x)$. [2]

(iv) Find an expression for $f^{-1}(x)$. [3]

Answer: $\frac{2}{3}\pi$; $2 \le f(x) \le 10$; $f^{-1}(x) = 2\cos^{-1}\left(\frac{6-x}{4}\right)$ **N14/12/Q11**

35 Functions f and g are defined by

$$f : x \mapsto 3x + 2, \quad x \in \mathbb{R},$$
$$g : x \mapsto 4x - 12, \quad x \in \mathbb{R}.$$

Solve the equation $f^{-1}(x) = gf(x)$. [4]

Answer: $x = \frac{2}{7}$ **N15/12/Q1**

36 The function f is defined, for $x \in \mathbb{R}$, by $f : x \mapsto x^2 + ax + b$, where a and b are constants.

(i) In the case where $a = 6$ and $b = -8$, find the range of f. [3]

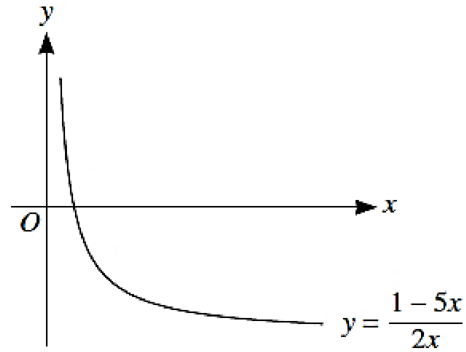
(ii) In the case where $a = 5$, the roots of the equation $f(x) = 0$ are k and $-2k$, where k is a constant. Find the values of b and k. [3]

(iii) Show that if the equation $f(x + a) = a$ has no real roots, then $a^2 < 4(b - a)$. [3]

Answers: **(i)** $f(x) \geqslant -17$ **(ii)** $k = 5$, $b = -50$ **N15/12/Q8**

Homework: Functions – Variants 11 & 13

1



The diagram shows the graph of $y = \mathrm{f}^{-1}(x)$, where f^{-1} is defined by $\mathrm{f}^{-1}(x) = \dfrac{1-5x}{2x}$ for $0 < x \leqslant 2$.

(i) Find an expression for $\mathrm{f}(x)$ and state the domain of f. [5]

(ii) The function g is defined by $\mathrm{g}(x) = \dfrac{1}{x}$ for $x \geqslant 1$. Find an expression for $\mathrm{f}^{-1}\mathrm{g}(x)$, giving your answer in the form $ax + b$, where a and b are constants to be found. [2]

Answers: (i) $\mathrm{f}(x) = \dfrac{1}{2x+5}$ for $x \geq -\dfrac{9}{4}$; (ii) $\dfrac{1}{2}x - \dfrac{5}{2}$. **13/J15/6**

---

2 The function $\mathrm{f} : x \mapsto 5 + 3\cos(\tfrac{1}{2}x)$ is defined for $0 \leqslant x \leqslant 2\pi$.

(i) Solve the equation $\mathrm{f}(x) = 7$, giving your answer correct to 2 decimal places. [3]

(ii) Sketch the graph of $y = \mathrm{f}(x)$. [2]

(iii) Explain why f has an inverse. [1]

(iv) Obtain an expression for $\mathrm{f}^{-1}(x)$. [3]

Answers: (i) 1.68 (iii) 1:1 mapping (iv) $2\cos^{-1}\left(\dfrac{x-5}{2}\right)$ **11/J15/8**

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3 (a) The functions f and g are defined for $x \geqslant 0$ by

$$\mathrm{f} : x \mapsto (ax+b)^{\frac{1}{3}}, \text{ where } a \text{ and } b \text{ are positive constants,}$$
$$\mathrm{g} : x \mapsto x^2.$$

Given that $\mathrm{fg}(1) = 2$ and $\mathrm{gf}(9) = 16$,

(i) calculate the values of a and b, [4]

(ii) obtain an expression for $\mathrm{f}^{-1}(x)$ and state the domain of f^{-1}. [4]

Answers: **(a)(i)** $a = 7$, $b = 1$; **(ii)** $f^{-1}(x) = \frac{1}{7}(x^3 - 1)$ for $x \geq 1$; 13/N14/10

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4 **(i)** Express $x^2 - 2x - 15$ in the form $(x + a)^2 + b$. [2]

The function f is defined for $p \leqslant x \leqslant q$, where p and q are positive constants, by

$$f : x \mapsto x^2 - 2x - 15.$$

The range of f is given by $c \leqslant f(x) \leqslant d$, where c and d are constants.

(ii) State the smallest possible value of c. [1]

For the case where $c = 9$ and $d = 65$,

(iii) find p and q, [4]

(iv) find an expression for $f^{-1}(x)$. [3]

Answers: **(i)** $(x - 1)^2 - 16$ **(ii)** -16 **(iii)** $p = 6$, $q = 10$ **(iv)** $f^{-1}(x) = 1 + \sqrt{x + 16}$ 11/N14/10

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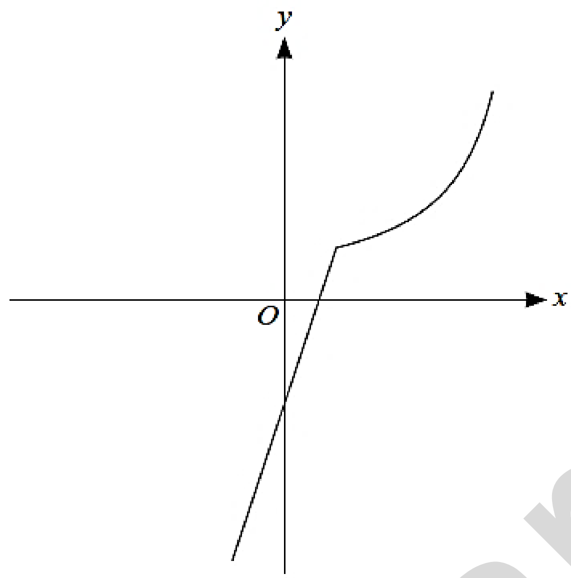
5 A function f is such that $f(x) = \dfrac{15}{2x + 3}$ for $0 \leqslant x \leqslant 6$.

(i) Find an expression for $f'(x)$ and use your result to explain why f has an inverse. [3]

(ii) Find an expression for $f^{-1}(x)$, and state the domain and range of f^{-1}. [4]

Answers: **(i)** $\dfrac{-30}{(2x+3)^2}$; **(ii)** Domain: $1 \leq x \leq 5$, range: $0 \leq f^{-1}(x) \leq 6$. 13/J14/5

6



The diagram shows the function f defined for $-1 \leqslant x \leqslant 4$, where

$$f(x) = \begin{cases} 3x - 2 & \text{for } -1 \leqslant x \leqslant 1, \\ \dfrac{4}{5-x} & \text{for } 1 < x \leqslant 4. \end{cases}$$

(i) State the range of f. [1]

(ii) Copy the diagram and on your copy sketch the graph of $y = f^{-1}(x)$. [2]

(iii) Obtain expressions to define the function f^{-1}, giving also the set of values for which each expression is valid. [6]

11/J14/10

Answers: (i) $-5 \le f(x) \le 4$; (iii) LINE: $f^{-1}(x) = \frac{1}{3}(x+2)$ for $-5 \le x \le 1$,

CURVE: $f^{-1}(x) = 5 - \frac{4}{x}$ for $1 < x \le 4$.

---

7 The function f is defined by $f : x \mapsto x^2 + 4x$ for $x \geqslant c$, where c is a constant. It is given that f is a one-one function.

(i) State the range of f in terms of c and find the smallest possible value of c. [3]

The function g is defined by $g : x \mapsto ax + b$ for $x \geqslant 0$, where a and b are positive constants. It is given that, when $c = 0$, $gf(1) = 11$ and $fg(1) = 21$.

(ii) Write down two equations in a and b and solve them to find the values of a and b. [6]

13/N13/10

Answer: i) $y \ge c^2 + 4c$ *and c is* -2 ii) *Answer*. $a = 2$, $b = 1$

8 The function f is defined by

$$f : x \mapsto x^2 + 1 \text{ for } x \geqslant 0.$$

(i) Define in a similar way the inverse function f^{-1}. [3]

(ii) Solve the equation $ff(x) = \frac{185}{16}$. [3]

Answers: (i) $f(x) = \sqrt{x-1}$ for $x \geq 1$; (ii) $x = 1.5$. **11/N13/5**

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9 The function f is defined by $f : x \mapsto 2x + k, x \in \mathbb{R}$, where k is a constant.

(i) In the case where $k = 3$, solve the equation $ff(x) = 25$. [2]

The function g is defined by $g : x \mapsto x^2 - 6x + 8, x \in \mathbb{R}$.

(ii) Find the set of values of k for which the equation $f(x) = g(x)$ has no real solutions. [3]

The function h is defined by $h : x \mapsto x^2 - 6x + 8, x > 3$.

(iii) Find an expression for $h^{-1}(x)$. [4]

Answers: (i) $x = 4$; (ii) $k < -8$; (iii) $3 + \sqrt{(1+x)}$. **13/J13/10**

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10 (i) Express $2x^2 - 12x + 13$ in the form $a(x+b)^2 + c$, where a, b and c are constants. [3]

(ii) The function f is defined by $f(x) = 2x^2 - 12x + 13$ for $x \geqslant k$, where k is a constant. It is given that f is a one-one function. State the smallest possible value of k. [1]

The value of k is now given to be 7.

(iii) Find the range of f. [1]

(iv) Find an expression for $f^{-1}(x)$ and state the domain of f^{-1}. [5]

Answers: (i) $2(x-3)^2 - 5$; (ii) 3; (iii) $y \geq 27$; (iv) $3 + \sqrt{\tfrac{1}{2}(x+5)}$ for $x \geq 27$. **11/J13/8**

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11 The functions f and g are defined for $-\frac{1}{2}\pi \leqslant x \leqslant \frac{1}{2}\pi$ by

$$f(x) = \tfrac{1}{2}x + \tfrac{1}{6}\pi,$$
$$g(x) = \cos x.$$

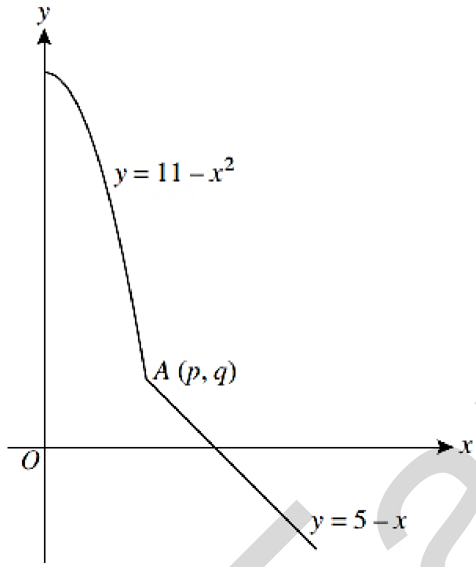
Solve the following equations for $-\frac{1}{2}\pi \leqslant x \leqslant \frac{1}{2}\pi$.

(i) $gf(x) = 1$, giving your answer in terms of π. [2]

(ii) $fg(x) = 1$, giving your answers correct to 2 decimal places. [4]

Answers: (i) $-\frac{\pi}{3}$; (ii) ± 0.31. 13/N12/6

12



(i) The diagram shows part of the curve $y = 11 - x^2$ and part of the straight line $y = 5 - x$ meeting at the point $A\ (p, q)$, where p and q are positive constants. Find the values of p and q. [3]

(ii) The function f is defined for the domain $x \geqslant 0$ by

$$f(x) = \begin{cases} 11 - x^2 & \text{for } 0 \leqslant x \leqslant p, \\ 5 - x & \text{for } x > p. \end{cases}$$

Express $f^{-1}(x)$ in a similar way. [5]

Answers: (i) $p=3$, $q=2$; (ii) $f^{-1}(x) = \begin{cases} 5 - x & x < 2 \\ \sqrt{11 - x}, & 2 \le x \le 11 \end{cases}$ 13/N12/7

13 The function f is defined by $f(x) = 4x^2 - 24x + 11$, for $x \in \mathbb{R}$.

(i) Express $f(x)$ in the form $a(x - b)^2 + c$ and hence state the coordinates of the vertex of the graph of $y = f(x)$. [4]

The function g is defined by $g(x) = 4x^2 - 24x + 11$, for $x \leqslant 1$.

(ii) State the range of g. [2]

(iii) Find an expression for $g^{-1}(x)$ and state the domain of g^{-1}. [4]

Answers: (i) $4(x-3)^2 - 25$, $(3, -25)$; (ii) $g(x) \ge -9$; (iii) $3 - 1/2\sqrt{x+25}$, $x \ge -9$. 11/N12/10

14 The function f is such that $f(x) = 8 - (x-2)^2$, for $x \in \mathbb{R}$.

(i) Find the coordinates and the nature of the stationary point on the curve $y = f(x)$. [3]

The function g is such that $g(x) = 8 - (x-2)^2$, for $k \leqslant x \leqslant 4$, where k is a constant.

(ii) State the smallest value of k for which g has an inverse. [1]

For this value of k,

(iii) find an expression for $g^{-1}(x)$, [3]

(iv) sketch, on the same diagram, the graphs of $y = g(x)$ and $y = g^{-1}(x)$. [3]

Answers: **(i)** (2,8), maximum **(ii)** $k = 2$ **(iii)** $g^{-1}(x) = 2 + \sqrt{(8 - x)}$ **(iv)** Sketch. 13/J12/11

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15 The function $f : x \mapsto x^2 - 4x + k$ is defined for the domain $x \geqslant p$, where k and p are constants.

(i) Express $f(x)$ in the form $(x + a)^2 + b + k$, where a and b are constants. [2]

(ii) State the range of f in terms of k. [1]

(iii) State the smallest value of p for which f is one-one. [1]

(iv) For the value of p found in part (iii), find an expression for $f^{-1}(x)$ and state the domain of f^{-1}, giving your answers in terms of k. [4]

Answers: **(i)** $(x-2)^2 - 4 + k$ **(ii)** $f(x) \geq k - 4$ **(iii)** 2 **(iv)** $f^{-1}(x) = 2 + \sqrt{x + 4 - k}$, $x \geq k - 4$ 11/J12/8

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16 Functions f and g are defined by

$$f : x \mapsto 2x + 3 \quad \text{for } x \leqslant 0,$$
$$g : x \mapsto x^2 - 6x \quad \text{for } x \leqslant 3.$$

(i) Express $f^{-1}(x)$ in terms of x and solve the equation $f(x) = f^{-1}(x)$. [3]

(ii) On the same diagram sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$, showing the coordinates of their point of intersection and the relationship between the graphs. [3]

(iii) Find the set of values of x which satisfy $gf(x) \leqslant 16$. [5]

13/N11/9

Answers: (i) $\frac{1}{2}(x-3), -3$, **(iii)** $-\frac{5}{2} \leq x \leq 0$

17 Functions f and g are defined by

$$f : x \mapsto 2x^2 - 8x + 10 \quad \text{for } 0 \leqslant x \leqslant 2,$$
$$g : x \mapsto x \quad \text{for } 0 \leqslant x \leqslant 10.$$

(i) Express $f(x)$ in the form $a(x+b)^2 + c$, where a, b and c are constants. [3]

(ii) State the range of f. [1]

(iii) State the domain of f^{-1}. [1]

(iv) Sketch on the same diagram the graphs of $y = f(x)$, $y = g(x)$ and $y = f^{-1}(x)$, making clear the relationship between the graphs. [4]

(v) Find an expression for $f^{-1}(x)$. [3]

Answers: **(i)** $2(x-2)^2+2$; **(ii)** $2 \le f(x) \le 10$; **(iii)** $2 \le x \le 10$; **(v)** $2-\sqrt{\frac{x-2}{2}}$. **11/N11/11**

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18 Functions f and g are defined by

$$f : x \mapsto 3x - 4, \quad x \in \mathbb{R},$$
$$g : x \mapsto 2(x-1)^3 + 8, \quad x > 1.$$

(i) Evaluate fg(2). [2]

(ii) Sketch in a single diagram the graphs of $y = f(x)$ and $y = f^{-1}(x)$, making clear the relationship between the graphs. [3]

(iii) Obtain an expression for $g'(x)$ and use your answer to explain why g has an inverse. [3]

(iv) Express each of $f^{-1}(x)$ and $g^{-1}(x)$ in terms of x. [4]

Answers: **(i)** 26; **(iii)** $6(x-1)^2$; **(iv)** $\frac{x+4}{3}$, $\sqrt[3]{\frac{x-8}{2}}+1$. **13/J11/10**

19 Functions f and g are defined for $x \in \mathbb{R}$ by

$$f : x \mapsto 2x + 1,$$
$$g : x \mapsto x^2 - 2.$$

(i) Find and simplify expressions for $fg(x)$ and $gf(x)$. [2]

(ii) Hence find the value of a for which $fg(a) = gf(a)$. [3]

(iii) Find the value of b $(b \neq a)$ for which $g(b) = b$. [2]

(iv) Find and simplify an expression for $f^{-1}g(x)$. [2]

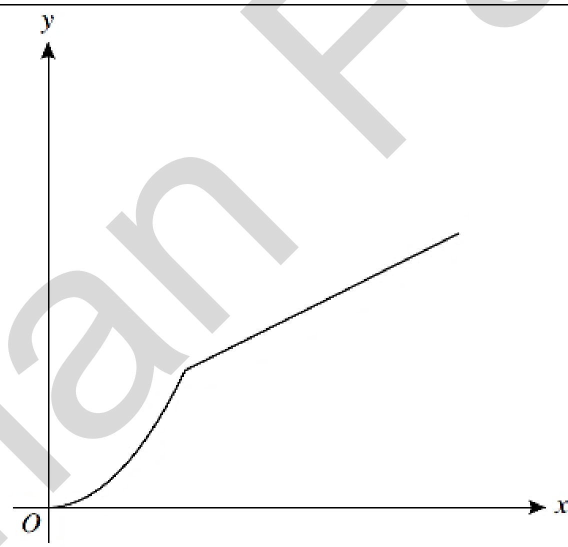
The function h is defined by

$$h : x \mapsto x^2 - 2, \quad \text{for } x \leqslant 0.$$

(v) Find an expression for $h^{-1}(x)$. [2]

Answers: (i) $2x^2 - 3$, $4x^2 + 4x - 1$; (ii) -1; (iii) 2; (iv) $\frac{1}{2}(x^2 - 3)$; (v) $-\sqrt{x + 2}$. 11/J11/11

20



The diagram shows the function f defined for $0 \leqslant x \leqslant 6$ by

$$x \mapsto \tfrac{1}{2}x^2 \quad \text{for } 0 \leqslant x \leqslant 2,$$
$$x \mapsto \tfrac{1}{2}x + 1 \quad \text{for } 2 < x \leqslant 6.$$

(i) State the range of f. [1]

(ii) Copy the diagram and on your copy sketch the graph of $y = f^{-1}(x)$. [2]

(iii) Obtain expressions to define $f^{-1}(x)$, giving the set of values of x for which each expression is valid. [4]

Answers: (i) $0 < f(x) < 4$; (iii) $x \mapsto \sqrt{2x}$ for $0 < x < 2$, $x \mapsto 2x - 2$ for $2 < x < 4$. 13/N10/7

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21 Functions f and g are defined for $x \in \mathbb{R}$ by

$$f : x \mapsto 2x + 3,$$
$$g : x \mapsto x^2 - 2x.$$

Express $gf(x)$ in the form $a(x + b)^2 + c$, where a, b and c are constants. [5]

Answer. $4(x + 1)^2 - 1$. 11/N10/3

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22 A function f is defined by $f : x \mapsto 3 - 2\tan(\frac{1}{2}x)$ for $0 \leqslant x < \pi$.

(i) State the range of f. [1]

(ii) State the exact value of $f(\frac{2}{3}\pi)$. [1]

(iii) Sketch the graph of $y = f(x)$. [2]

(iv) Obtain an expression, in terms of x, for $f^{-1}(x)$. [3]

Answers: (i) $f(x) < 3$; (ii) $3 - 2\sqrt{3}$; (iv) $2\tan^{-1}\left(\frac{3-x}{2}\right)$. 11/N10/7

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23 The function $f : x \mapsto a + b\cos x$ is defined for $0 \leqslant x \leqslant 2\pi$. Given that $f(0) = 10$ and that $f(\frac{2}{3}\pi) = 1$, find

(i) the values of a and b, [2]

(ii) the range of f, [1]

(iii) the exact value of $f(\frac{5}{6}\pi)$. [2]

Answers: (i) 4, 6; (ii) $-2 \leqslant f(x) \leqslant 10$; (iii) $4 - 3\sqrt{3}$. 13/J10/3

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24 The function $f : x \mapsto 2x^2 - 8x + 14$ is defined for $x \in \mathbb{R}$.

(i) Find the values of the constant k for which the line $y + kx = 12$ is a tangent to the curve $y = f(x)$. [4]

(ii) Express $f(x)$ in the form $a(x + b)^2 + c$, where a, b and c are constants. [3]

(iii) Find the range of f. [1]

The function $g : x \mapsto 2x^2 - 8x + 14$ is defined for $x \geqslant A$.

(iv) Find the smallest value of A for which g has an inverse. [1]

(v) For this value of A, find an expression for $g^{-1}(x)$ in terms of x. [3]

13/J10/10

Answers: (i) 4 or 12; (ii) $2(x-2)^2+6$; (iii) $f(x) \geqslant 6$; (iv) 2; (v) $\sqrt{\frac{(x-6)}{2}}+2$.

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25 The function f is such that $f(x) = 2\sin^2 x - 3\cos^2 x$ for $0 \leqslant x \leqslant \pi$.

(i) Express $f(x)$ in the form $a + b\cos^2 x$, stating the values of a and b. [2]

(ii) State the greatest and least values of $f(x)$. [2]

(iii) Solve the equation $f(x) + 1 = 0$. [3]

Answers: (i) 2, –5; (ii) 2, –3; (iii) 0.685, 2.46. 11/J10/5

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25 The function f is defined by $f : x \mapsto 2x^2 - 12x + 7$ for $x \in \mathbb{R}$.

(i) Express $f(x)$ in the form $a(x-b)^2 - c$. [3]

(ii) State the range of f. [1]

(iii) Find the set of values of x for which $f(x) < 21$. [3]

The function g is defined by $g : x \mapsto 2x + k$ for $x \in \mathbb{R}$.

(iv) Find the value of the constant k for which the equation $gf(x) = 0$ has two equal roots. [4]

Answers: (i) $2(x-3)^2-11$; (ii) $f(x) \geqslant -11$; (iii) $-1 < x < 7$; (iv) 22. 11/J10/9

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26 (i) Express $-x^2 + 6x - 5$ in the form $a(x+b)^2 + c$, where a, b and c are constants. [3]

The function $f : x \mapsto -x^2 + 6x - 5$ is defined for $x \geqslant m$, where m is a constant.

(ii) State the smallest value of m for which f is one-one. [1]

(iii) For the case where $m = 5$, find an expression for $f^{-1}(x)$ and state the domain of f^{-1}. [4]

N15/11/Q9

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27 The function f is defined by $f(x) = 3x + 1$ for $x \leqslant a$, where a is a constant. The function g is defined by $g(x) = -1 - x^2$ for $x \leqslant -1$.

(i) Find the largest value of a for which the composite function gf can be formed. [2]

For the case where $a = -1$,

(ii) solve the equation $fg(x) + 14 = 0$, [3]

(iii) find the set of values of x which satisfy the inequality $gf(x) \leqslant -50$. [4]

N15/11/Q8

28 (a) Find the values of the constant m for which the line $y = mx$ is a tangent to the curve $y = 2x^2 - 4x + 8$. [3]

(b) The function f is defined for $x \in \mathbb{R}$ by $f(x) = x^2 + ax + b$, where a and b are constants. The solutions of the equation $f(x) = 0$ are $x = 1$ and $x = 9$. Find

(i) the values of a and b, [2]

(ii) the coordinates of the vertex of the curve $y = f(x)$. [2]

Answer: **(a)** $m = -12,\ m = 4$ **(b)(i)** $a = -10,\ b = 9$ **(ii)** $(5,\ -16)$

J16/11/Q6

29 The function f is defined by $f : x \mapsto 4\sin x - 1$ for $-\frac{1}{2}\pi \leqslant x \leqslant \frac{1}{2}\pi$.

(i) State the range of f. [2]

(ii) Find the coordinates of the points at which the curve $y = f(x)$ intersects the coordinate axes. [3]

(iii) Sketch the graph of $y = f(x)$. [2]

(iv) Obtain an expression for $f^{-1}(x)$, stating both the domain and range of f^{-1}. [4]

Answers: **(i)** $-5 \leqslant x \leqslant 3$ **(ii)** $(0,-1)$, $(0.253,0)$ **(iv)** $f^{-1}(x) = \sin^{-1}\frac{x+1}{4}$, $-5 \le x \le 3$, $\frac{-\pi}{2} \le f^{-1}(x) \le \frac{\pi}{2}$

J16/11/Q11

30 The function f is such that $f(x) = 2x + 3$ for $x \geqslant 0$. The function g is such that $g(x) = ax^2 + b$ for $x \leqslant q$, where a, b and q are constants. The function fg is such that $fg(x) = 6x^2 - 21$ for $x \leqslant q$.

(i) Find the values of a and b. [3]

(ii) Find the greatest possible value of q. [2]

It is now given that $q = -3$.

(iii) Find the range of fg. [1]

(iv) Find an expression for $(fg)^{-1}(x)$ and state the domain of $(fg)^{-1}$. [3]

Answers: **(i)** $a = 3$, $b = 12$; **(ii)** maximum $q = 2$; **(iii)** $y \ge 33$; **(iv)** $(fg)^{-1}(x) = -\sqrt{\frac{x+21}{6}}$, $x \ge 33$.

J16/13/Q10

31 The function f is defined by $f : x \mapsto \dfrac{2}{3-2x}$ for $x \in \mathbb{R}$, $x \neq \frac{3}{2}$.

(i) Find an expression for $f^{-1}(x)$. [3]

The function g is defined by $g : x \mapsto 4x + a$ for $x \in \mathbb{R}$, where a is a constant.

(ii) Find the value of a for which $gf(-1) = 3$. [3]

(iii) Find the possible values of a given that the equation $f^{-1}(x) = g^{-1}(x)$ has two equal roots. [4]

Answers: **(i)** $f^{-1}(x) = \dfrac{(3x-2)}{2x}$ or equivalent **(ii)** $\dfrac{7}{5}$ **(iii)** -10 and -2 J17/11/Q9

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32 (i) Express $9x^2 - 6x + 6$ in the form $(ax + b)^2 + c$, where a, b and c are constants. [3]

The function f is defined by $f(x) = 9x^2 - 6x + 6$ for $x \geqslant p$, where p is a constant.

(ii) State the smallest value of p for which f is a one-one function. [1]

15

(iii) For this value of p, obtain an expression for $f^{-1}(x)$, and state the domain of f^{-1}. [4]

(iv) State the set of values of q for which the equation $f(x) = q$ has no solution. [1]

Answers : **(i)** $(3x-1)^2 - 5$; **(ii)** Smallest value of p is 1 / 3; **(iii)** $f^{-1}(x) = \dfrac{\sqrt{x-5}+1}{3}$ for $x \geqslant 5$; **(iv)** $q < 5$. J17/13/Q9

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33 Functions f and g are defined for $x \in \mathbb{R}$ by

$$f : x \mapsto \tfrac{1}{2}x - 2,$$
$$g : x \mapsto 4 + x - \tfrac{1}{2}x^2.$$

(i) Find the points of intersection of the graphs of $y = f(x)$ and $y = g(x)$. [3]

(ii) Find the set of values of x for which $f(x) > g(x)$. [2]

17

(iii) Find an expression for $fg(x)$ and deduce the range of fg. [4]

The function h is defined by $h : x \mapsto 4 + x - \frac{1}{2}x^2$ for $x \geqslant k$.

(iv) Find the smallest value of k for which h has an inverse. [2]

Answers: **(i)** (4.0) *and* $(-3,-3.5)$ **(ii)** $x < -3, x > 4$ **(iii)** $fg(x) \leqslant \dfrac{1}{4}$ **(iv)** $k = 1$ J18/11/Q9

34 The one-one function f is defined by $f(x) = (x-2)^2 + 2$ for $x \geqslant c$, where c is a constant.

(i) State the smallest possible value of c. [1]

In parts (ii) and (iii) the value of c is 4.

(ii) Find an expression for $f^{-1}(x)$ and state the domain of f^{-1}. [3]

17

(iii) Solve the equation $ff(x) = 51$, giving your answer in the form $a + \sqrt{b}$. [5]

Answers: **(i)** Smallest value of c is 2; **(ii)** $f^{-1}(x) = \sqrt{x-2} + 2$; **(iii)** $x = 2 + \sqrt{7}$. **J18/13/Q10**

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35 The functions f and g are defined by

$$f(x) = \frac{4}{x} - 2 \quad \text{for } x > 0,$$

$$g(x) = \frac{4}{5x+2} \quad \text{for } x \geqslant 0.$$

(i) Find and simplify an expression for $fg(x)$ and state the range of fg. [3]

(ii) Find an expression for $g^{-1}(x)$ and find the domain of g^{-1}. [5]

Answers: **(i)** $fg(x) = 5x$, $fg(x) \geqslant 0$ **(ii)** $g^{-1}(x) = (4-2x)/5x, 0 < x \leqslant 2$ **N16/11/Q8**

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36 (i) Express $4x^2 + 12x + 10$ in the form $(ax+b)^2 + c$, where a, b and c are constants. [3]

(ii) Functions f and g are both defined for $x > 0$. It is given that $f(x) = x^2 + 1$ and $fg(x) = 4x^2 + 12x + 10$. Find $g(x)$. [1]

(iii) Find $(fg)^{-1}(x)$ and give the domain of $(fg)^{-1}$. [4]

Answers: **(i)** $(2x+3)^2 + 1$; **(ii)** $2x + 3$; **(iii)** $[\sqrt{(x-1)} - 3]/2$, for $x > 10$. **N16/13/Q8**

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37 Functions f and g are defined for $x > 3$ by

$$f : x \mapsto \frac{1}{x^2 - 9},$$

$$g : x \mapsto 2x - 3.$$

(i) Find and simplify an expression for $gg(x)$. [2]

(ii) Find an expression for $f^{-1}(x)$ and state the domain of f^{-1}. [4]

(iii) Solve the equation $fg(x) = \frac{1}{7}$. [4]

Answers: **(i)** $4x-9$ **(ii)** $\sqrt{\frac{1}{x}+9}$, $x>0$ **(iii)** $x=\frac{7}{2}$

N17/11/Q9

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38 The functions f and g are defined by

$$f(x) = \frac{2}{x^2-1} \text{ for } x < -1,$$
$$g(x) = x^2 + 1 \text{ for } x > 0.$$

(i) Find an expression for $f^{-1}(x)$. [3]

(ii) Solve the equation $gf(x) = 5$. [4]

Answers: **(i)** $-\sqrt{\frac{2}{x}+1}$ **(ii)** $-\sqrt{2}$

N17/13/Q6

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39 (a) The one-one function f is defined by $f(x) = (x-3)^2 - 1$ for $x < a$, where a is a constant.

(i) State the greatest possible value of a. [1]

(ii) It is given that a takes this greatest possible value. State the range of f and find an expression for $f^{-1}(x)$. [3]

(b) The function g is defined by $g(x) = (x-3)^2$ for $x \geqslant 0$.

(i) Show that $gg(2x)$ can be expressed in the form $(2x-3)^4 + b(2x-3)^2 + c$, where b and c are constants to be found. [2]

(ii) Hence expand $gg(2x)$ completely, simplifying your answer. [4]

Answers: **(a)(i)** 3 **(ii)** $y > -1$, $f^{-1}(x) = 3 - \sqrt{1+x}$ **(b)(i)** $(2x-3)^4 - 6(2x-3)^2 + 9$
(ii) $16x^4 - 96x^3 + 192x^2 - 144x + 36$

N18/11/Q11

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40 (i) Express $2x^2 - 12x + 11$ in the form $a(x+b)^2 + c$, where a, b and c are constants. [3]

The function f is defined by $f(x) = 2x^2 - 12x + 11$ for $x \leqslant k$.

(ii) State the largest value of the constant k for which f is a one-one function. [1]

(iii) For this value of k find an expression for $f^{-1}(x)$ and state the domain of f^{-1}. [4]

The function g is defined by $g(x) = x + 3$ for $x \leqslant p$.

(iv) With k now taking the value 1, find the largest value of the constant p which allows the composite function fg to be formed, and find an expression for $fg(x)$ whenever this composite function exists. [3]

N18/13/Q11

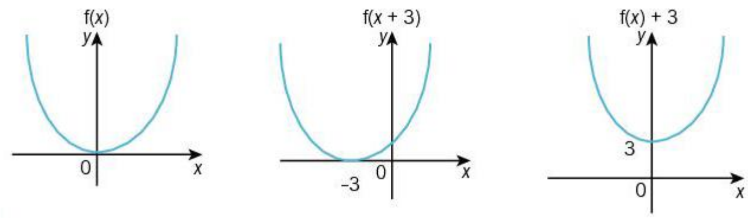
Answers: **(i)** $2(x-3)^2-7$ **(ii)** Largest value of k is 3 **(iii)** $f^{-1}(x)=3-\sqrt{\frac{1}{2}(x+7)}$ $x \geqslant -7$

(iv) largest p is -2, $fg(x)=2x^2-7$

Transformation of functions

You can transform the graph of a function by moving it horizontally or vertically.
This **transformation** is called a **translation**.

> $f(x + a)$ is a horizontal translation of $-a$.
> $f(x) + a$ is a vertical translation of a.



Note: We can write $y = f(x)$, $y = f(x + 3)$ and $y = f(x) + 3$ for these graphs.

Example 1

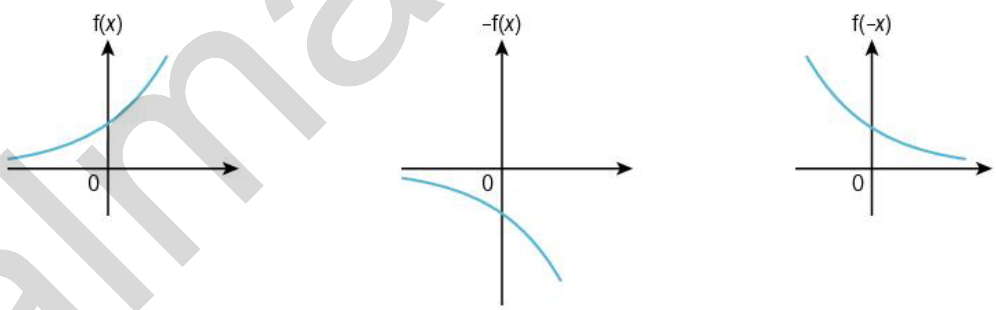
$f(x) = x^3$ $\qquad$ $g(x) = \dfrac{1}{a}$

Sketch the graphs of the following functions, marking on each sketch where the curve cuts the axes and state the equations of any **asymptotes**:

a) $f(x - 2)$ $\qquad$ **b)** $g(x) - 2$

You can transform the graph of a function by **reflecting** the graph in one of the axes.

> $-f(x)$ is a reflection in the x-axis.
> $f(-x)$ is a reflection in the y-axis.



Example 2

$f(x) = x(x + 1)(x - 2)$

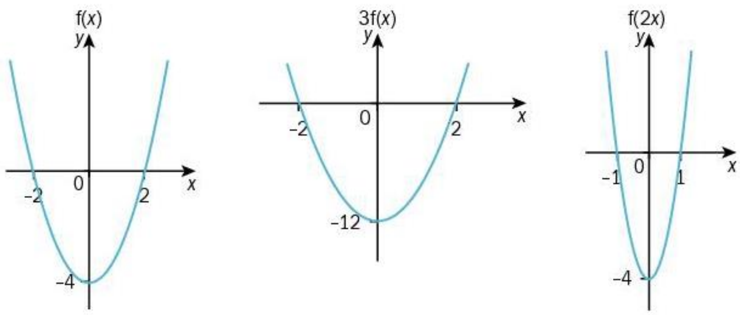
Sketch the graphs of the following functions

a) $1 + f(-x)$ $\qquad$ **b)** $-f(x + 3)$.

You can transform the graph of a function by **stretching** (or compressing) the graph horizontally or vertically.

$af(x)$ is a stretch with factor a in the y-direction.
$f(ax)$ is a stretch with factor $\frac{1}{a}$ in the x-direction.

Note: $af(x)$ means **multiply** all the y-values by a while the x-values stay the same.
$f(ax)$ means **divide** all the x-values by a while the y-values stay the same.

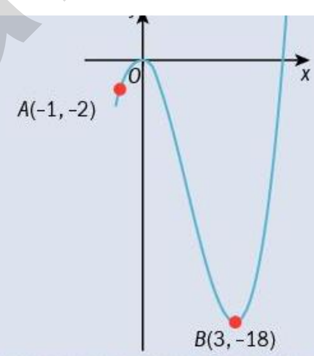


Example 3

The diagram shows a sketch of the curve $f(x)$, which passes through the origin, O, and the points $A(-1, -2)$ and $B(3, -18)$.
Sketch the graphs of

a) $-2f(x)$ **b)** $f(-3x)$

In each case, mark the new position of the points O, A and B, writing down their coordinates.



Homework: Transformation of Functions Practice

1

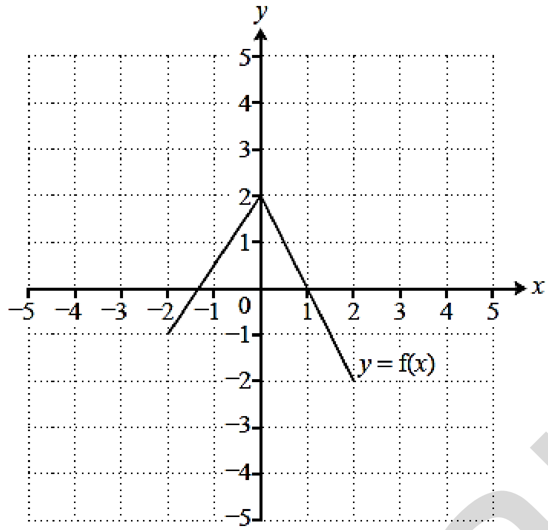


Fig. 3

Fig. 3 shows the graph of $y = f(x)$. Draw the graphs of the following.

(i) $y = f(x) - 2$ **[2]**

(ii) $y = f(x - 3)$ **[2]**

Answer: i) Vertices at (-2,-3), (0,0) and (2,-4)
ii) Vertices at (1,-1), (3,2) and (5,-2)

C1 Curve Sketching
Transformations 1
Question 1

2

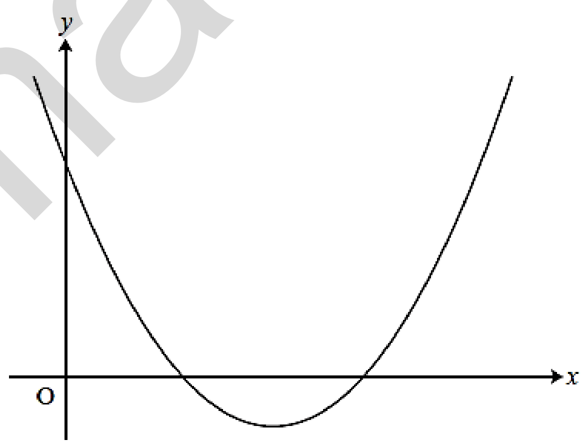


Fig. 11

Fig. 11 shows a sketch of the curve with equation $y = (x-4)^2 - 3$.

(i) Write down the equation of the line of symmetry of the curve and the coordinates of the minimum point. **[2]**

(ii) Find the coordinates of the points of intersection of the curve with the x-axis and the y-axis, using surds where necessary. **[4]**

(iii) The curve is translated by $\begin{pmatrix}2\\0\end{pmatrix}$. Show that the equation of the translated curve may be written as $y = x^2 - 12x + 33$. **[2]**

(iv) Show that the line $y = 8 - 2x$ meets the curve $y = x^2 - 12x + 33$ at just one point, and find the coordinates of this point. **[5]**

Answer: i) x=4 and (4,-3) ii) (0,13) $[x=]4 \pm \sqrt{3}$ or $\frac{8 \pm \sqrt{12}}{2}$ **and iii) (5,-2)**

C1 Curve Sketching Transformations 1 Question 2

3

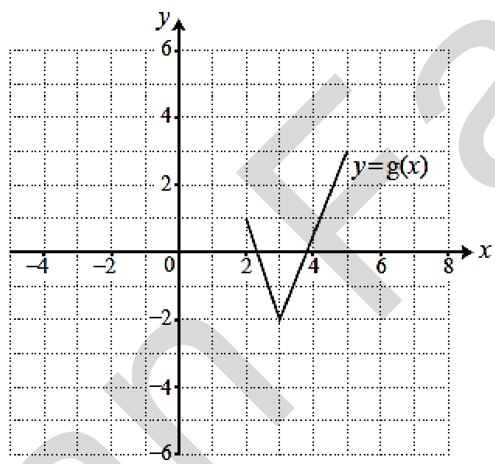


Fig. 7

Fig. 7 shows the graph of $y = \text{g}(x)$. Draw the graphs of the following.

(i) $y = \text{g}(x) + 3$ **[2]**

(ii) $y = \text{g}(x + 2)$ **[2]**

Answer: i) tick at (2,4), (3,1) and (5,6) ii) tick at (0,1), (1,-2) and (3,3)

C1 Curve Sketching Transformations 1 Question 4

4 The point P (5, 4) is on the curve $y = \text{f}(x)$. State the coordinates of the image of P when the graph of $y = \text{f}(x)$ is transformed to the graph of

(i) $y = \text{f}(x - 5)$, **[2]**

(ii) $y = \text{f}(x) + 7$. **[2]**

Answer: i) (10,4) ii) (5,4)

C1 Curve Sketching Transformations 1 Question 5

5 (i) Describe fully the transformation which maps the curve $y = x^2$ onto the curve $y = (x+4)^2$. [2]

(ii) Sketch the graph of $y = x^2 - 4$. [2]

Answer: i) Translation by $\begin{pmatrix} -4 \\ 0 \end{pmatrix}$ or 4 [units] to left

ii) min at (0,-4) and graph through -2 and 2 on x-axis

C1 Curve Sketching Transformations 1 Question 6

6 (i) A curve has equation $y = x^2 - 4$. Find the x-coordinates of the points on the curve where $y = 21$. [2]

(ii) The curve $y = x^2 - 4$ is translated by $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$.

Write down an equation for the translated curve. You need not simplify your answer. [2]

Answer: i) x = ±5 ii) $y = (x-2)^2 - 4$ or $y = x^2 - 4x$

C1 Curve Sketching Transformations 2 Question 1

7

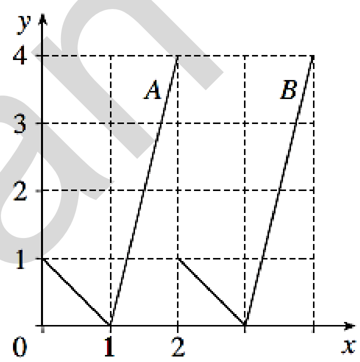


Fig. 2

Fig. 2 shows graphs A and B.

(i) State the transformation which maps graph A onto graph B. [2]

(ii) The equation of graph A is $y = f(x)$.

Which one of the following is the equation of graph B?

$y = f(x) + 2$	$y = f(x) - 2$	$y = f(x+2)$	$y = f(x-2)$
$y = 2f(x)$	$y = f(x+3)$	$y = f(x-3)$	$y = 3f(x)$

[2]

Answer: i) Translation of $\binom{2}{0}$ ii) $y = f(x-2)$

C1 Curve Sketching Transformations 2 Question 2

8 You are given that $f(x) = (x+3)(x-2)(x-5)$.

(i) Sketch the curve $y = f(x)$. [3]

(ii) Show that $f(x)$ may be written as $x^3 - 4x^2 - 11x + 30$. [2]

(iii) Describe fully the transformation that maps the graph of $y = f(x)$ onto the graph of $y = g(x)$, where $g(x) = x^3 - 4x^2 - 11x - 6$. [2]

Answer: i) crossing x-axis at -3,2 and 5, crossing y-axis at 30

ii) Translation of $\binom{0}{-36}$

C1 Curve Sketching Transformations 2 Question 3

9 (i) Sketch the graph of $y = 3\sqrt{x}$, for $x \geqslant 0$. [1]

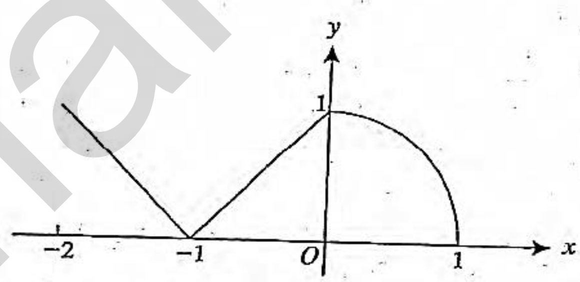
(ii) The graph of $y = 3\sqrt{x}$ is stretched by a factor of 2 parallel to the y-axis. State the equation of the transformed graph. [2]

(iii) Describe the single geometrical transformation that transforms the graph of $y = 3\sqrt{x}$ to the graph of $y = 3\sqrt{(x-k)}$. [2]

Answer: ii) $y = 6\sqrt{x}$ iii) Translation of $\binom{k}{0}$

C1 transformation of Graphs Question 1

10



The diagram shows the graph of $y = f(x)$ for $-2 \leqslant x \leqslant 1$. Outside this interval $f(x)$ is zero.

Sketch, on separate diagrams, the graphs of

(i) $y = f(x+1)$, [2]

(ii) $y = -3f(x)$. [2]

Label each graph in the same way as in the diagram above.

C1 Transformation of Graphs Question 2

11 (i) Sketch the curve $y = x^3$. [1]

(ii) Describe a transformation that transforms the curve $y = x^3$ to the curve $y = -x^3$. [2]

(iii) The curve $y = x^3$ is translated by p units, parallel to the x-axis. State the equation of the curve after it has been transformed. [2]

Answer: ii) Reflection in the x-axis iii) $y = (x - p)^3$ **C1 Transformation of Graphs Question 3**

12 (i) Given that $f(x) = x^2$, sketch the graph of $y = f(x)$. [1]

The graph of $y = g(x)$ is obtained by reflecting the graph of $y = f(x)$ in the x-axis. The graph of $y = h(x)$ is obtained by translating the graph of $y = g(x)$ by +2 units parallel to the y-axis.

(ii) Sketch and label the graphs of $y = g(x)$ and $y = h(x)$ on a single diagram. [3]

(iii) Write down expressions for $g(x)$ and $h(x)$ in terms of x. [2]

Answer: iii) $g(x) = -x^2$, $h(x) = -x^2 + 2$ **C1 Transformation of Graphs Question 4**

13

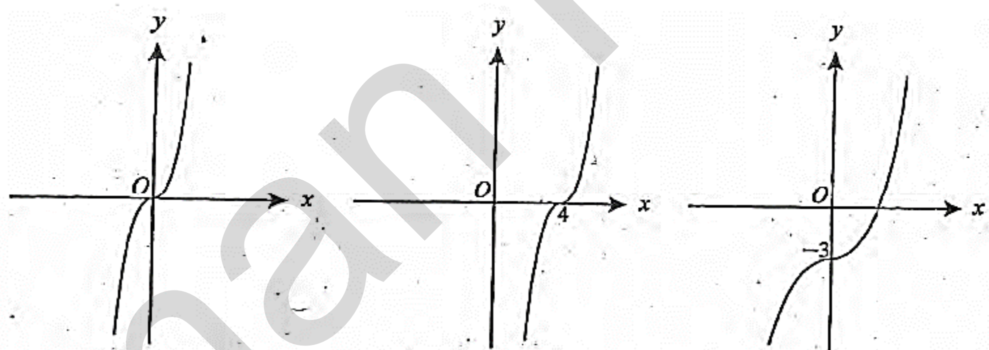


Diagram A Diagram B Diagram C

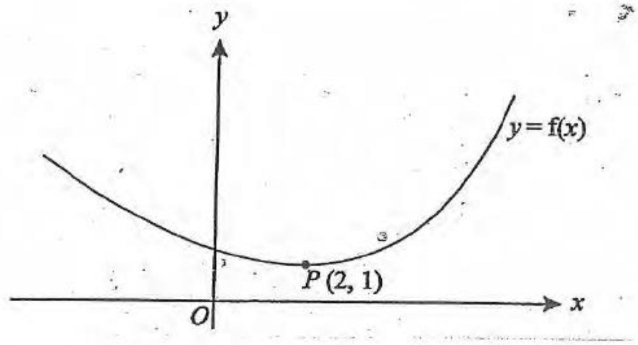
Diagram A shows the graph of $y = x^3$.

(i) The graph of $y = x^3$ is given a translation of +4 units parallel to the x-axis, as shown in diagram B. Write down the equation of the transformed graph. [2]

(ii) The graph of $y = x^3$ is given a stretch parallel to the x-axis with factor 2, followed by a translation of −3 units parallel to the y-axis, as shown in diagram C. Write down the equation of the transformed graph. [3]

Answer: i) $y = (x - 4)^3$ $ii)\ \frac{1}{8}x^3 - 3$ **C1 Transformation of Graphs Question 5**

14



The diagram shows the graph of $y = f(x)$. The point $P\,(2,\ 1)$ lies on the curve.

(i) Sketch, on separate diagrams, the following graphs. On each graph label the image of the point P, giving its coordinates.

(a) $y = -f(x)$. [1]

(b) $y = 2f(x+3)$. [3]

(ii) The graph of $y = 2f(x+3)$ is obtained from the graph of $y = f(x)$ by a sequence of two geometrical transformations. Describe each of these transformations fully. [2]

Answer: i) $P(2,-1)$ *and* $P(-1,2)$

ii) Translation of $\binom{-3}{0}$ and stretch sf 2 in y-direction

C1 Transformation of Graphs Question 6

15 The point R $(6, -3)$ is on the curve $y = f(x)$.

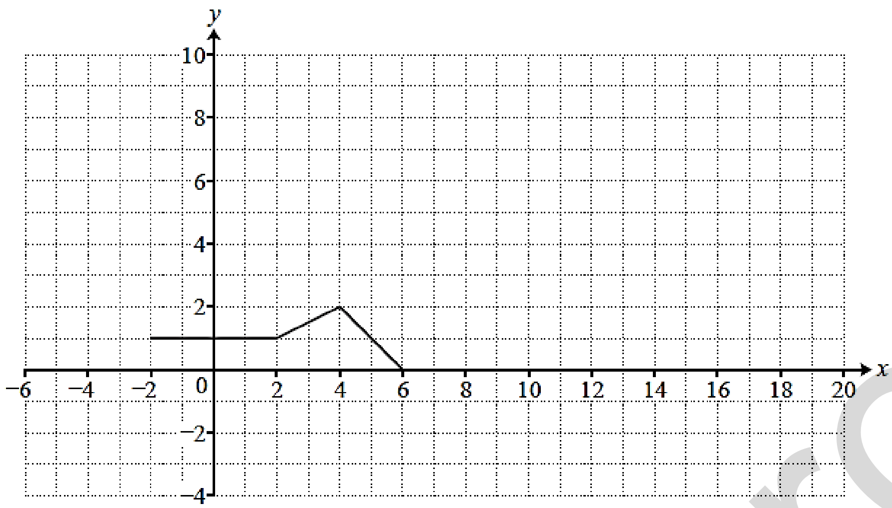
(i) Find the coordinates of the image of R when the curve is transformed to $y = \frac{1}{2}f(x)$. [2]

(ii) Find the coordinates of the image of R when the curve is transformed to $y = f(3x)$. [2]

Answer: i) (6,-1.5) ii) (2,-3)

C2 Curve Sketching Question 1

16



Draw the graph of

(i) $y = g(2x)$, **[2]**

(ii) $y = 3g(x)$. **[2]**

Answer: i) graph from (-1, 1) to (1, 1) to (2, 2) to (3, 0)

ii) graph from (-2, 3) to (2, 3) to (4, 6) to (6, 0)

C2 Curve Sketching Question 2

17 The point P (6, 3) lies on the curve $y = f(x)$. State the coordinates of the image of P after the transformation which maps $y = f(x)$ onto

(i) $y = 3f(x)$, **[2]**

(ii) $y = f(4x)$. **[2]**

Answer: i) (6,9) ii) (1.5,3)

C2 Curve Sketching Question 3

18 In this question, $f(x) = x^2 - 5x$. Fig. 4 shows a sketch of the graph of $y = f(x)$.

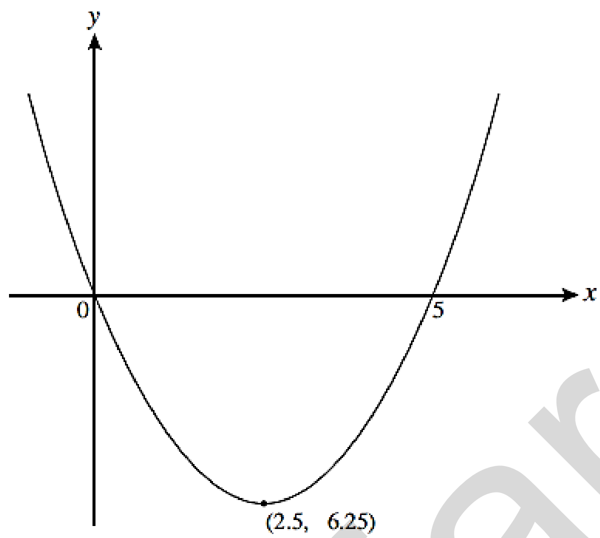


Fig. 4

On separate diagrams, sketch the curves $y = f(2x)$ and $y = 3f(x)$, labelling the coordinates of their intersections with the axes and their turning points. [4]

Answer: i) crossing *x*-axis at 0 and 2.5 min at (1.25, −6.25),
crossing *x*-axis at 0 and 5 min at (2.5, −18.75)

C2 Curve Sketching
Question 4

19 State the transformation which maps the graph of $y = x^2 + 5$ onto the graph of $y = 3x^2 + 15$. [2]

Answer: i) stretch, parallel to the *y* axis, sf 3

C2 Curve Sketching
Question 5

20

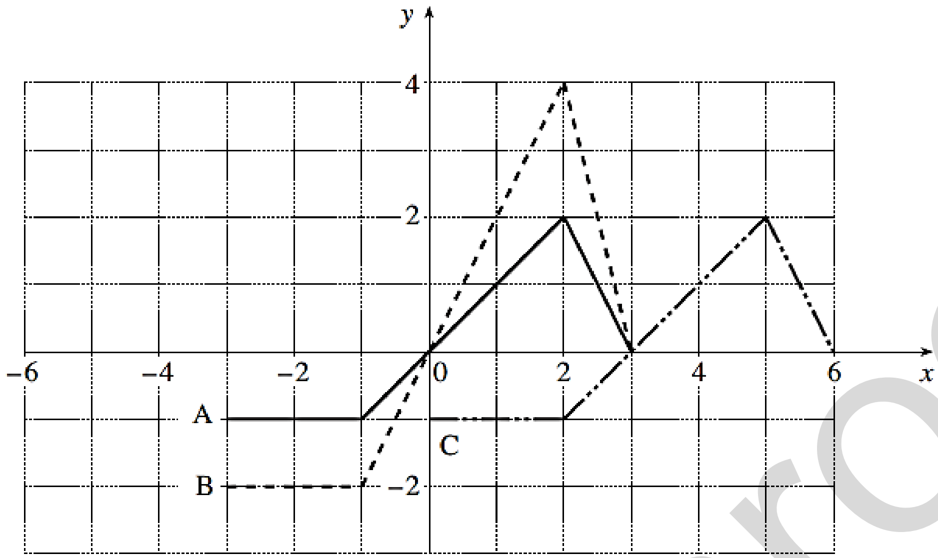


Fig. 3

Fig. 3 shows sketches of three graphs, A, B and C. The equation of graph A is $y = f(x)$.

State the equation of

(i) graph B, [2]

(ii) graph C. [2]

Answer: i) $y = 2f(x)$ **ii)** $y = f(x - 3)$

C2 Curve Sketching Question 6

21

(i) The point P (4, –2) lies on the curve $y = f(x)$. Find the coordinates of the image of P when the curve is transformed to $y = f(5x)$. [2]

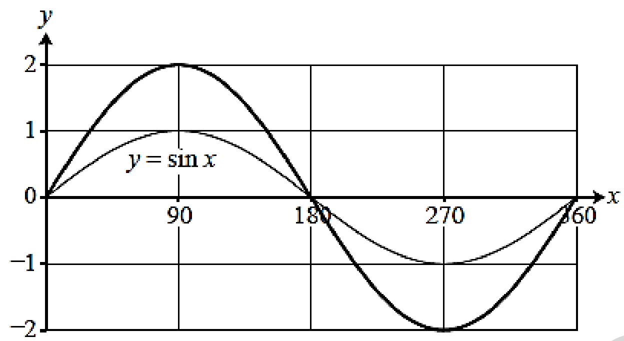
(ii) Describe fully a single transformation which maps the curve $y = \sin x°$ onto the curve $y = \sin(x - 90)°$. [2]

Answer: i) $(0.8, -2)$ **ii) Translation of** $\binom{90}{0}$

C2 Curve Sketching Question 8

22 Figs. 5.1 and 5.2 show the graph of $y = \sin x$ for values of x from 0° to 360° and two transformations of this graph. State the equation of each graph after it has been transformed.

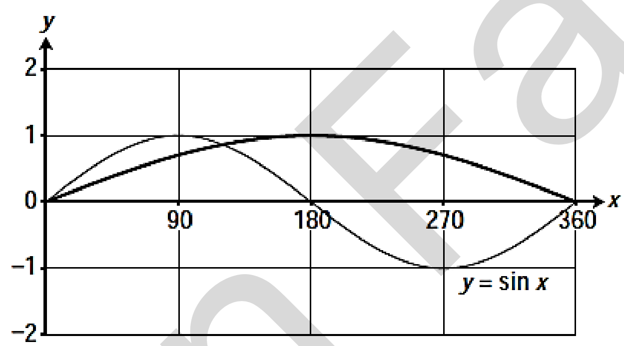
(i)



[1]

Fig. 5.1

(ii)



[2]

Fig. 5.2

Answer: i) $y = 2\sin(x)$ **ii)** $y = \sin(0.5x)$

C2 Curve Sketching Question 9

23 The curve $y = f(x)$ has a minimum point at (3, 5).

State the coordinates of the corresponding minimum point on the graph of

(i) $y = 3f(x)$, **[2]**

(ii) $y = f(2x)$. **[2]**

Answer: i) (3,15) ii) (1.5,5)

C2 Curve Sketching Question 10

24

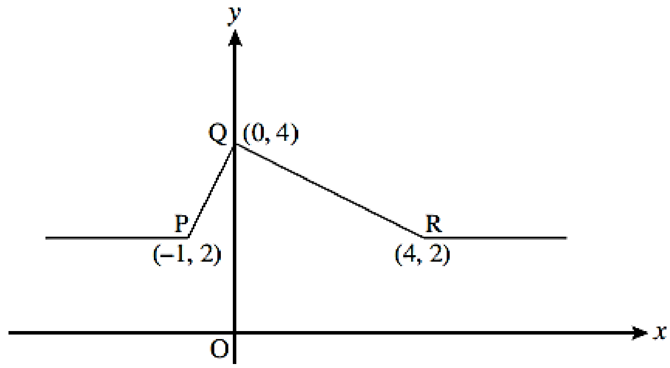


Fig. 5

Fig. 5 shows a sketch of the graph of $y = f(x)$. On separate diagrams, sketch the graphs of the following, showing clearly the coordinates of the points corresponding to P, Q and R.

(i) $y = f(2x)$ **[2]**

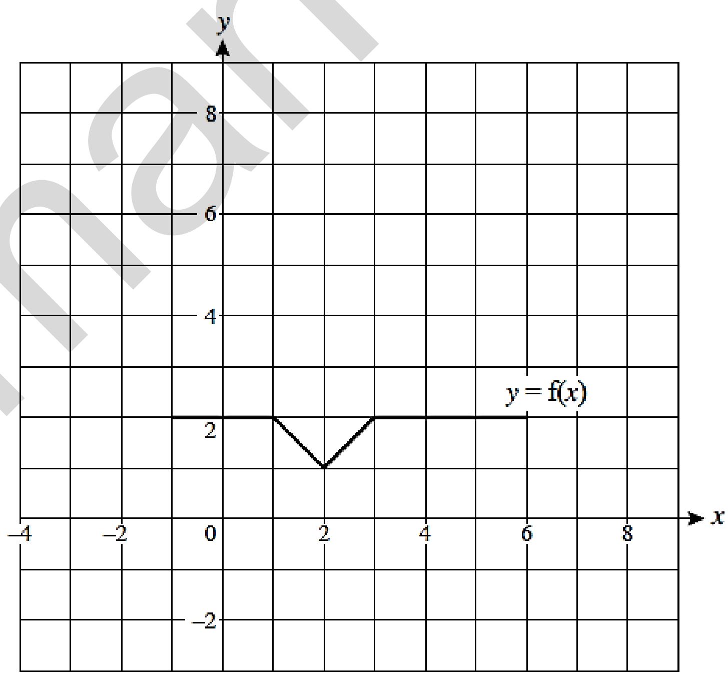
(ii) $y = \frac{1}{4}f(x)$ **[2]**

Answer: i) P (-0.5,2), Q (0,4) and R (2,2)

ii) P (-1,0.5), Q (0,1) and R (4,0.5)

C2 Curve Sketching
Question 11

25 Fig. 5 shows the graph of $y = f(x)$.



On the insert, draw the graph of

(i) $y = f(x-2)$, [2]

(ii) $y = 3f(x)$. [2]

Answer: i) graph along y = 2 with V at (3,2) (4,1) & (5,2)

ii) graph along y = 6 with V at (1,6) (2,3) & (3,6)

C2 Curve Sketching Question 12

26

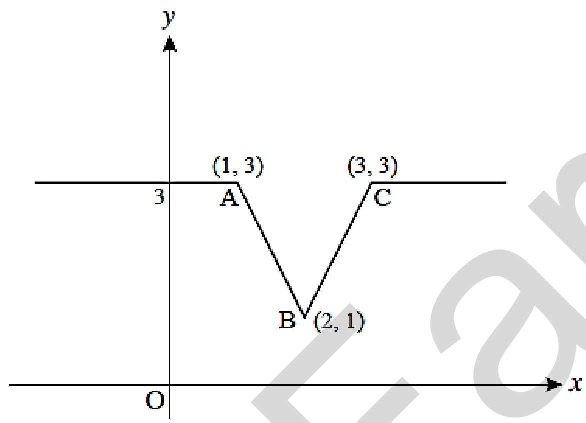


Fig. 4

Fig. 4 shows a sketch of the graph of $y = f(x)$. On separate diagrams, sketch the graphs of the following, showing clearly the coordinates of the points corresponding to A, B and C.

(i) $y = 2f(x)$ [2]

(ii) $y = f(x+3)$ [2]

Answer: i) line along y = 6 with V (1, 6), (2, 2), (3, 6)

ii) line along y = 3 with V (−2,3), (−1,1), (0,3)

C2 Curve Sketching Question 13

27 (i) On the same axes, sketch the graphs of $y = \cos x$ and $y = \cos 2x$ for values of x from 0 to 2π. [3]

(ii) Describe the transformation which maps the graph of $y = \cos x$ onto the graph of $y = 3\cos x$. [2]

Answer: ii) (1-way) stretch parallel to y axis sf 3

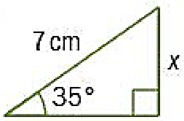
C2 Curve Sketching Question 14

Trigonometry

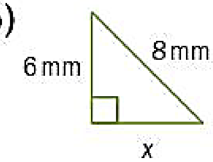
Example 1

Find the side or angle marked x.

a)



b)



c)

Example 2

a) Convert to radians

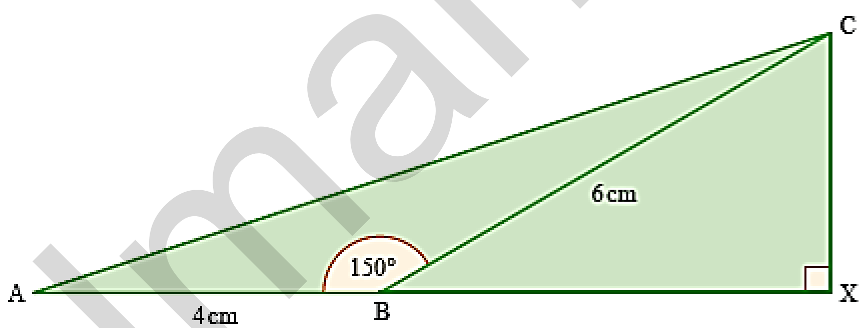
i) $90°$ **ii)** $225°$ **iii)** $43°$.

b) Convert to degrees

i) $\frac{3\pi}{4}$ rad **ii)** $\frac{7\pi}{5}$ rad **iii)** 2.5 rad.

Example 3

In the diagram, ABC is a triangle in which AB = 4 cm, BC = 6 cm and angle ABC = 150°. The line CX is perpendicular to the line ABX.



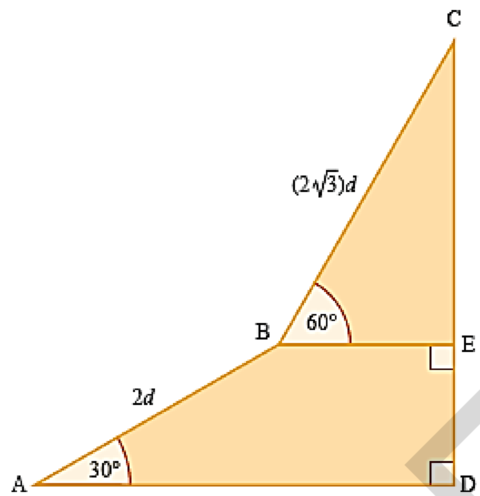
(i) Find the exact length of BX and show that angle CAB $= \tan^{-1}\left(\frac{3}{4+3\sqrt{3}}\right)$

(ii) Show that the exact length of AC is $\sqrt{(52 + 24\sqrt{3})}$ cm.

[Cambridge AS & A Level Mathematics 9709, Paper 1 Q6 June 2006]

Example 4

In the diagram, ABED is a trapezium with right angles at E and D, and CED is a straight line. The lengths of AB and BC are $2d$ and $(2\sqrt{3})d$ respectively, and angles BAD and CBE are 30° and 60° respectively.



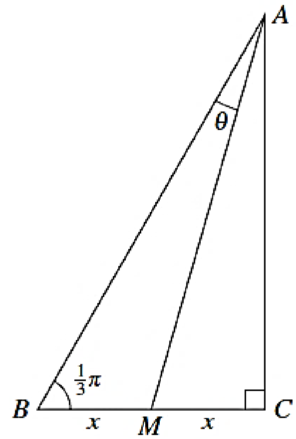
(i) Find the length of CD in terms of d.

(ii) Show that angle CAD = $\tan^{-1}\left(\frac{2}{\sqrt{3}}\right)$

[**Cambridge AS & A Level Mathematics 9709, Paper 1 Q3 November 2005**]

Example 5

June 2016/12 Question 5



In the diagram, triangle ABC is right-angled at C and M is the mid-point of BC. It is given that angle $ABC = \frac{1}{3}\pi$ radians and angle $BAM = \theta$ radians. Denoting the lengths of BM and MC by x,

(i) find AM in terms of x, [3]

(ii) show that $\theta = \frac{1}{6}\pi - \tan^{-1}\left(\frac{1}{2\sqrt{3}}\right)$. [2]

Example 6

Write down the value of

i) $\sin\left(\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right)$ **ii)** $\sin\left(\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right)$ **iii)** $\tan(\tan^{-1}1)$ **iv)** $\cos\left(\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)\right)$.

Is $\sin(\sin^{-1}(x)) = x$ for all values of x?

Example 7

November 2014/11 Question 2

Find the value of x satisfying the equation $\sin^{-1}(x-1) = \tan^{-1}(3)$. [3]

Example 8

Find all the angles x (to the nearest degree), where $0° < x < 360°$, such that $\sin x = 0.88$.

Example 9

Solve $\tan x = 2.35$ for $0° < x < 720°$.
Give your answers correct to one decimal place.

Example 10

Find values of θ in the interval $-360° \leqslant \theta \leqslant 360°$ for which $\sin\theta = 0.5$.

Example 11

Solve the equation $3\tan\theta = -1$ for $-180° \leqslant \theta \leqslant 180°$.

Example 12

Solve the equation $4\sin x + 3 = 5$ for $0° \leq x \leq 360°$.

Example 13

Solve the equation $3\cos 2\theta = -1$ for $0° \leq \theta \leq 360°$.

Example 12

Find values of θ in the interval $0° \leqslant \theta \leqslant 360°$ for which $\tan^2\theta - \tan\theta = 2$.

Example 13

Solve the equation $2\sin^2\theta = \cos\theta + 1$ for $0° \leqslant \theta \leqslant 360°$.

Example 14

Simplify **a)** $\dfrac{\sin^3\theta + \sin\theta\cos^2\theta}{\cos\theta}$ **b)** $\dfrac{1+\cos\theta}{\sin\theta} + \dfrac{\sin\theta}{1+\cos\theta}$.

Example 15

Prove the identity $\dfrac{\sin\theta}{\tan\theta(1-\cos\theta)} = \dfrac{\cos\theta}{1-\cos\theta}$,

Example 16

Prove the identity $\dfrac{1-\tan^2 x}{1+\tan^2 x} \equiv 1 - 2\sin^2 x$.

[Cambridge AS & A Level Mathematics 9709, Paper 1 Q3 June 2007]

Example 17

Prove the identity $\dfrac{1+\sin x}{\cos x} + \dfrac{\cos x}{1+\sin x} \equiv \dfrac{2}{\cos x}$

[Cambridge AS & A Level Mathematics 9709, Paper 1 Q2 November 2008]

Example 18

Prove the identity $\dfrac{\sin x}{1-\sin x} - \dfrac{\sin x}{1+\sin x} \equiv 2\tan^2 x$.

[Cambridge AS & A Level Mathematics 9709, Paper 1 Q1 June 2009]

Example 19

Solve $\sqrt{3}\sin x - \cos x = 0$ for $0 \le x \le 2\pi$.

Example 20

Solve the equation $2\cos^2 x = 1 - \sin x$ for $0° \le x \le 360°$.

Example 21

Solve the equation $\sin^2(2x - 60°) = 0.6$ for $0° \le x \le 180°$.

Example 22

(i) Show that the equation $3(2\sin x - \cos x) = 2(\sin x - 3\cos x)$ can be written in the form $\tan x = -\frac{3}{4}$.

(ii) Solve the equation $3(2\sin x - \cos x) = 2(\sin x - 3\cos x)$, for $0° \leqslant x \leqslant 360°$.

[Cambridge AS & A Level Mathematics 9709, Paper 12 Q1 June 2010]

Example 23

(i) Prove the identity $(\sin x + \cos x)(1 - \sin x \cos x) \equiv \sin^3 x + \cos^3 x$.

(ii) Solve the equation $(\sin x + \cos x)(1 - \sin x \cos x) = 9\sin^3 x$ for $0° \leqslant x \leqslant 360°$.

[Cambridge AS & A Level Mathematics 9709, Paper 12 Q5 November 2009]

Example 24

(i) Show that the equation $\sin\theta + \cos\theta = 2(\sin\theta - \cos\theta)$ can be expressed as $\tan\theta = 3$.

(ii) Hence solve the equation $\sin\theta + \cos\theta = 2(\sin\theta - \cos\theta)$, for $0° \leqslant \theta \leqslant 360°$

[Cambridge AS & A Level Mathematics 9709, Paper 1 Q3 June 2005]

Example 25

Solve the equation $3\sin^2\theta - 2\cos\theta - 3 = 0$, for $0° \leqslant x \leqslant 180°$.

[Cambridge AS & A Level Mathematics 9709, Paper 1 Q1 November 2005]

Example 26

June 2015/12 Question 5

(i) Prove the identity $\dfrac{\sin\theta - \cos\theta}{\sin\theta + \cos\theta} \equiv \dfrac{\tan\theta - 1}{\tan\theta + 1}$. [1]

(ii) Hence solve the equation $\dfrac{\sin\theta - \cos\theta}{\sin\theta + \cos\theta} = \dfrac{\tan\theta}{6}$, for $0° \leqslant \theta \leqslant 180°$. [4]

Example 27

June 2015/13 Question 4

(i) Express the equation $3\sin\theta = \cos\theta$ in the form $\tan\theta = k$ and solve the equation for $0° < \theta < 180°$. [2]

(ii) Solve the equation $3\sin^2 2x = \cos^2 2x$ for $0° < x < 180°$. [4]

Example 28

November 2014/11 Question 3

Solve the equation $\dfrac{13\sin^2\theta}{2+\cos\theta} + \cos\theta = 2$ for $0° \leqslant \theta \leqslant 180°$. [4]

Example 29

November 2014/12 Question 5

(i) Show that the equation $1 + \sin x \tan x = 5\cos x$ can be expressed as

$$6\cos^2 x - \cos x - 1 = 0.$$ [3]

(ii) Hence solve the equation $1 + \sin x \tan x = 5\cos x$ for $0° \leqslant x \leqslant 180°$. [3]

Example 30

November 2014/13 Question 5

(i) Show that $\sin^4\theta - \cos^4\theta \equiv 2\sin^2\theta - 1$. [3]

(ii) Hence solve the equation $\sin^4\theta - \cos^4\theta = \frac{1}{2}$ for $0° \leqslant \theta \leqslant 360°$. [4]

Example 31

2020 Specimen Paper 1 Question 7

(a) Show that the equation $1 + \sin x \tan x = 5\cos x$ can be expressed as

$$6\cos^2 x - \cos x - 1 = 0.$$ [3]

(b) Hence solve the equation $1 + \sin x \tan x = 5\cos x$ for $0° \leqslant x \leqslant 180°$. [3]

Example 32

November 2016/11 Question 6

(i) Show that $\cos^4 x \equiv 1 - 2\sin^2 x + \sin^4 x$. [1]

(ii) Hence, or otherwise, solve the equation $8\sin^4 x + \cos^4 x = 2\cos^2 x$ for $0° \leqslant x \leqslant 360°$. [5]

Example 33

November 2016/12 Question 2

(i) Express the equation $\sin 2x + 3\cos 2x = 3(\sin 2x - \cos 2x)$ in the form $\tan 2x = k$, where k is a constant. [2]

(ii) Hence solve the equation for $-90° \leqslant x \leqslant 90°$. [3]

Example 34

November 2016/13 Question 3

Showing all necessary working, solve the equation $6\sin^2 x - 5\cos^2 x = 2\sin^2 x + \cos^2 x$ for $0° \leqslant x \leqslant 360°$. [4]

Example 35

June 2016/11 Question 2

Solve the equation $3\sin^2\theta = 4\cos\theta - 1$ for $0° \leqslant \theta \leqslant 360°$. [4]

Example 36

June 2016/12 Question 7

(i) Prove the identity $\dfrac{1+\cos\theta}{1-\cos\theta} - \dfrac{1-\cos\theta}{1+\cos\theta} \equiv \dfrac{4}{\sin\theta\tan\theta}$. [4]

(ii) Hence solve, for $0° < \theta < 360°$, the equation

$$\sin\theta\left(\frac{1+\cos\theta}{1-\cos\theta} - \frac{1-\cos\theta}{1+\cos\theta}\right) = 3.$$ [3]

Example 37

June 2016/13 Question 8

(i) Show that $3\sin x\tan x - \cos x + 1 = 0$ can be written as a quadratic equation in $\cos x$ and hence solve the equation $3\sin x\tan x - \cos x + 1 = 0$ for $0 \leqslant x \leqslant \pi$. [5]

(ii) Find the solutions to the equation $3\sin 2x\tan 2x - \cos 2x + 1 = 0$ for $0 \leqslant x \leqslant \pi$. [3]

Example 38

March 2016/12 Question 4

(a) Solve the equation $\sin^{-1}(3x) = -\frac{1}{3}\pi$, giving the solution in an exact form. [2]

(b) Solve, by factorising, the equation $2\cos\theta\sin\theta - 2\cos\theta - \sin\theta + 1 = 0$ for $0 \leqslant \theta \leqslant \pi$. [4]

Example 39

November 2015/11 Question 4

(i) Show that the equation $\frac{4\cos\theta}{\tan\theta} + 15 = 0$ can be expressed as

$$4\sin^2\theta - 15\sin\theta - 4 = 0.$$ [3]

(ii) Hence solve the equation $\frac{4\cos\theta}{\tan\theta} + 15 = 0$ for $0° \leqslant \theta \leqslant 360°$. [3]

Example 40

November 2015/12 Question 4

(i) Prove the identity $\left(\frac{1}{\sin x} - \frac{1}{\tan x}\right)^2 \equiv \frac{1-\cos x}{1+\cos x}$. [4]

(ii) Hence solve the equation $\left(\frac{1}{\sin x} - \frac{1}{\tan x}\right)^2 = \frac{2}{5}$ for $0 \leqslant x \leqslant 2\pi$. [3]

Example 41

June 2014/12 Question 3

The reflex angle θ is such that $\cos\theta = k$, where $0 < k < 1$.

(i) Find an expression, in terms of k, for

(a) $\sin\theta$, [2]

(b) $\tan\theta$. [1]

(ii) Explain why $\sin 2\theta$ is negative for $0 < k < 1$. [2]

Example 42

June 2015/11 Question 1

Given that θ is an obtuse angle measured in radians and that $\sin\theta = k$, find, in terms of k, an expression for

(i) $\cos\theta$, [1]

(ii) $\tan\theta$, [2]

(iii) $\sin(\theta + \pi)$. [1]

Example 43

Sketch the graph of $y = 3\cos x$ for $0° \leq x \leq 360°$.

Example 44

Sketch the graph of $y = \sin 2x$ for $0 \le x \le 2\pi$.

Starting with the curve $y = \cos x$, show how transformations can be used to sketch these curves.

(i) $y = \cos 3x$

(ii) $y = 3 + \cos x$

(iii) $y = \cos(x - 60°)$

(iv) $y = 2\cos x$

Example 45

(i) The function $f : x \mapsto a + b\sin x$ is defined for $0 \leqslant x \leqslant 2\pi$.

Given that $f(0) = 4$ and $f\left(\frac{\pi}{6}\right) = 5$,

(a) find the values of a and b

(b) the range of f

(c) sketch the graph of $y = a + b\sin x$ for $0 \leqslant x \leqslant 2\pi$.

(ii) The function $g : x \mapsto a + b\sin x$, where a and b have the same value as found in part **(i)** is defined for the domain $\frac{\pi}{2} \leqslant x \leqslant k$. Find the largest value of k for which $g(x)$ has an inverse.

Example 46

The function f is defined by $f(x) = a + b\cos 2x$, for $0 \leqslant x \leqslant \pi$. It is given that $f(0) = -1$ and $f\left(\frac{1}{2}\pi\right) = 7$.

(i) Find the values of a and b.

(ii) Find the x co-ordinates of the points where the curve $y = f(x)$ intersects the x axis.

(iii) Sketch the graph of $y = f(x)$.

[Cambridge AS & A Level Mathematics 9709, Paper 1 Q8 June 2007]

Example 47

The function f is such that $f(x) = a - b\cos x$ for $0° \leqslant x \leqslant 360°$, where a and b are positive constants. The maximum value of $f(x)$ is 10 and the minimum value is -2.

(i) Find the values of a and b.

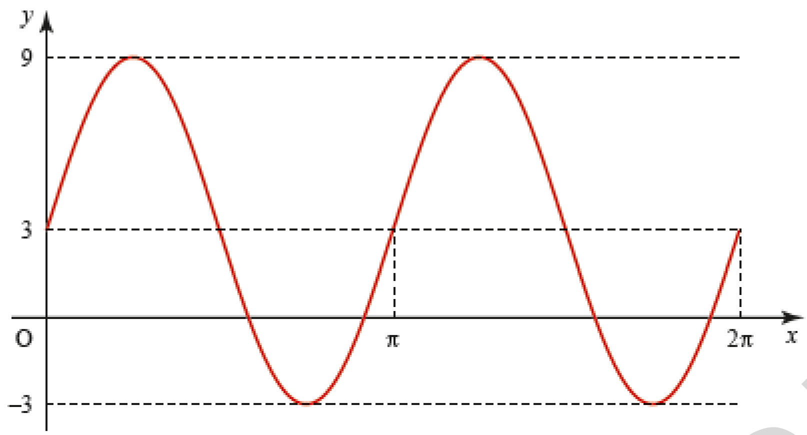
(ii) Solve the equation $f(x) = 0$.

(iii) Sketch the graph of $y = f(x)$.

[Cambridge AS & A Level Mathematics 9709, Paper 1 Q5 November 2008]

Example 48

The diagram shows the graph of $y = a\sin(bx) + c$ for $0 \leqslant x \leqslant 2\pi$.



(i) Find the values of a, b and c.

(ii) Find the smallest value of x in the interval $0 \leqslant x \leqslant 2\pi$ for which $y = 0$.

[Cambridge AS & A Level Mathematics 9709, Paper 1 Q4 June 2009]

Example 49

The function f is defined by $f : x \mapsto 5 - 3\sin 2x$ for $0 \leqslant x \leqslant \pi$.

(i) Find the range of f.

(ii) Sketch the graph of $y = f(x)$.

(iii) State, with a reason, whether f has an inverse.

[Cambridge AS & A Level Mathematics 9709, Paper 12 Q4 November 2009]

Example 50

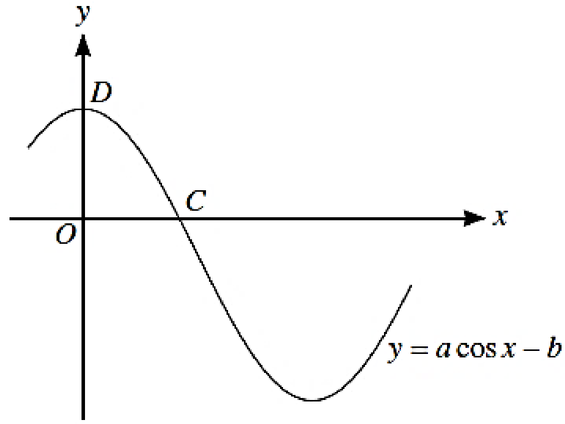
November 2015/13 Question 7

(a) Show that the equation $\dfrac{1}{\cos\theta} + 3\sin\theta\tan\theta + 4 = 0$ can be expressed as

$$3\cos^2\theta - 4\cos\theta - 4 = 0,$$

and hence solve the equation $\dfrac{1}{\cos\theta} + 3\sin\theta\tan\theta + 4 = 0$ for $0° \leqslant \theta \leqslant 360°$. [6]

(b)



The diagram shows part of the graph of $y = a\cos x - b$, where a and b are constants. The graph crosses the x-axis at the point $C(\cos^{-1}c,\ 0)$ and the y-axis at the point $D(0,\ d)$. Find c and d in terms of a and b. [2]

Homework: Trigonometry Variant 12

1 Find all the values of x in the interval $0° \leqslant x \leqslant 180°$ which satisfy the equation $\sin 3x + 2\cos 3x = 0$. [4]

Answers: 38.9°, 98.9°, 158.9°. J03/Q2

2 **(i)** Sketch the graph of the curve $y = 3\sin x$, for $-\pi \leqslant x \leqslant \pi$. [2]

The straight line $y = kx$, where k is a constant, passes through the maximum point of this curve for $-\pi \leqslant x \leqslant \pi$.

(ii) Find the value of k in terms of π. [2]

(iii) State the coordinates of the other point, apart from the origin, where the line and the curve intersect. [1]

Answers: **(i)** Sketch; **(ii)** $k = \frac{6}{\pi}$; **(iii)** $(-\frac{1}{2}\pi, -3)$. J03/Q6

3 **(i)** Show that the equation $\sin^2\theta + 3\sin\theta\cos\theta = 4\cos^2\theta$ can be written as a quadratic equation in $\tan\theta$. [2]

(ii) Hence, or otherwise, solve the equation in part **(i)** for $0° \leqslant \theta \leqslant 180°$. [3]

Answers: (i) $\tan^2\theta + 3\tan\theta - 4 = 0$; (ii) 45°, 104.0°. J04/Q3

4 **(i)** Show that the equation $\sin\theta + \cos\theta = 2(\sin\theta - \cos\theta)$ can be expressed as $\tan\theta = 3$. [2]

(ii) Hence solve the equation $\sin\theta + \cos\theta = 2(\sin\theta - \cos\theta)$, for $0° \leqslant \theta \leqslant 360°$. [2]

Answers: (ii) 71.6° and 251.6°. J05/Q3

5 Solve the equation

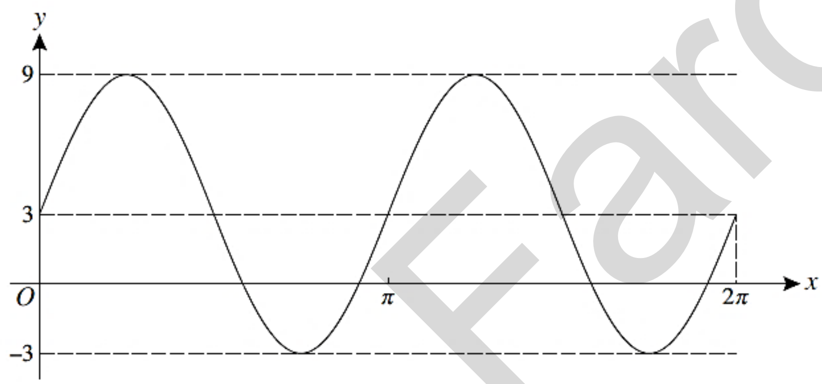
$$\sin 2x + 3\cos 2x = 0,$$

for $0° \leqslant x \leqslant 180°$. [4]

Answers: 54.2°, 144.2°. J06/Q2

7 Prove the identity $\frac{1 - \tan^2 x}{1 + \tan^2 x} \equiv 1 - 2\sin^2 x$. [4]

J07/Q3

8	**(i)** Show that the equation $2\tan^2\theta\cos\theta = 3$ can be written in the form $2\cos^2\theta + 3\cos\theta - 2 = 0$. [2] **(ii)** Hence solve the equation $2\tan^2\theta\cos\theta = 3$, for $0° \leqslant \theta \leqslant 360°$. [3]	
	Answer: (ii) 60°, 300°.	J08/Q2
9	Prove the identity $\dfrac{\sin x}{1-\sin x} - \dfrac{\sin x}{1+\sin x} \equiv 2\tan^2 x$. [3]	
	Answer: Proof	J09/Q1
10	 The diagram shows the graph of $y = a\sin(bx) + c$ for $0 \leqslant x \leqslant 2\pi$. **(i)** Find the values of a, b and c. [3] **(ii)** Find the smallest value of x in the interval $0 \leqslant x \leqslant 2\pi$ for which $y = 0$. [3]	
	Answers: (i) $a = 6$, $b = 2$, $c = 3$; (ii) $\dfrac{7\pi}{12}$ or 1.83	J09/Q4
11	**(i)** Show that the equation $$3(2\sin x - \cos x) = 2(\sin x - 3\cos x)$$ can be written in the form $\tan x = -\frac{3}{4}$. [2] **(ii)** Solve the equation $3(2\sin x - \cos x) = 2(\sin x - 3\cos x)$, for $0° \leqslant x \leqslant 360°$. [2]	
	Answer: 143.1°, 323.1°.	J10/12/Q1

12	(i) Show that the equation $3\tan\theta = 2\cos\theta$ can be expressed as $$2\sin^2\theta + 3\sin\theta - 2 = 0.$$ [3] (ii) Hence solve the equation $3\tan\theta = 2\cos\theta$, for $0° \leqslant \theta \leqslant 360°$. [3]
	Answer: (i) Proof; (ii) 30°, 150°. **N02/Q5**
13	(i) Show that the equation $4\sin^4\theta + 5 = 7\cos^2\theta$ may be written in the form $4x^2 + 7x - 2 = 0$, where $x = \sin^2\theta$. [1] (ii) Hence solve the equation $4\sin^4\theta + 5 = 7\cos^2\theta$, for $0° \leqslant \theta \leqslant 360°$. [4]
	Answers: (ii) 30°, 150°, 210°, 330°. **N03/Q2**
14	(i) Sketch and label, on the same diagram, the graphs of $y = 2\sin x$ and $y = \cos 2x$, for the interval $0 \leqslant x \leqslant \pi$. [4] (ii) Hence state the number of solutions of the equation $2\sin x = \cos 2x$ in the interval $0 \leqslant x \leqslant \pi$. [1]
	Answers: (i) Sketches; (ii) 2. **N04/Q4**
15	Solve the equation $3\sin^2\theta - 2\cos\theta - 3 = 0$, for $0° \leqslant \theta \leqslant 180°$. [4]
	Answers: 90°, 131.8°. **N05/Q1**
16	Given that $x = \sin^{-1}\left(\frac{2}{5}\right)$, find the exact value of (i) $\cos^2 x$, [2] (ii) $\tan^2 x$. [2]
	Answer: (i) $\frac{21}{25}$; (ii) $\frac{4}{21}$. **N06/Q2**
17	(i) Show that the equation $3\sin x\tan x = 8$ can be written as $3\cos^2 x + 8\cos x - 3 = 0$. [3] (ii) Hence solve the equation $3\sin x\tan x = 8$ for $0° \leqslant x \leqslant 360°$. [3]
	Answers: (ii) 70.5°, 289.5°. **N07/Q5**
18	Prove the identity $$\frac{1+\sin x}{\cos x} + \frac{\cos x}{1+\sin x} \equiv \frac{2}{\cos x}.$$ [4]

N08/Q2

19 **(i)** Prove the identity $(\sin x + \cos x)(1 - \sin x \cos x) \equiv \sin^3 x + \cos^3 x$. [3]

(ii) Solve the equation $(\sin x + \cos x)(1 - \sin x \cos x) = 9 \sin^3 x$ for $0° \leqslant x \leqslant 360°$. [3]

N09/12/Q5

Answer: **(ii)** 26.6°, 206.6°.

20 Prove the identity

$$\tan^2 x - \sin^2 x \equiv \tan^2 x \sin^2 x.$$ [4]

N10/12/Q2

21 **(i)** Prove the identity $\dfrac{\cos\theta}{\tan\theta(1-\sin\theta)} \equiv 1 + \dfrac{1}{\sin\theta}$. [3]

(ii) Hence solve the equation $\dfrac{\cos\theta}{\tan\theta(1-\sin\theta)} = 4$, for $0° \leqslant \theta \leqslant 360°$. [3]

Answer: **(ii)** 19.5°, 160.5°.

J11/12/Q5

22 The function f is such that $f(x) = 3 - 4\cos^k x$, for $0 \leqslant x \leqslant \pi$, where k is a constant.

(i) In the case where $k = 2$,

(a) find the range of f, [2]

(b) find the exact solutions of the equation $f(x) = 1$. [3]

(ii) In the case where $k = 1$,

(a) sketch the graph of $y = f(x)$, [2]

(b) state, with a reason, whether f has an inverse. [1]

Answers: **(a)(i)** $-1 \leqslant f(x) \leqslant 3$; **(ii)** $\frac{1}{4}\pi$, $\frac{3}{4}\pi$; **(b)(ii)** f has an inverse, f is one-one or increasing in the domain.

J11/12/Q9

23 **(i)** Sketch, on the same diagram, the graphs of $y = \sin x$ and $y = \cos 2x$ for $0° \leqslant x \leqslant 180°$. [3]

(ii) Verify that $x = 30°$ is a root of the equation $\sin x = \cos 2x$, and state the other root of this equation for which $0° \leqslant x \leqslant 180°$. [2]

(iii) Hence state the set of values of x, for $0° \leqslant x \leqslant 180°$, for which $\sin x < \cos 2x$. [2]

Answers: **(i)** Sketch; **(ii)** 150°; **(iii)** $0° \leq x < 30°$, $150° < x \leq 180°$

N11/12/Q5

24 (i) Prove the identity $\tan x + \frac{1}{\tan x} \equiv \frac{1}{\sin x \cos x}$. [2]

(ii) Solve the equation $\frac{2}{\sin x \cos x} = 1 + 3\tan x$, for $0° \leqslant x \leqslant 180°$. [4]

Answers: **(i)** Proof **(ii)** x = 45° or 116.6° **J12/12/Q5**

25 It is given that $a = \sin\theta - 3\cos\theta$ and $b = 3\sin\theta + \cos\theta$, where $0° \leqslant \theta \leqslant 360°$.

(i) Show that $a^2 + b^2$ has a constant value for all values of θ. [3]

(ii) Find the values of θ for which $2a = b$. [4]

Answers: **(i)** 10. **(ii)** 98.1°, 278.1°. **J13/12/Q5**

26 The reflex angle θ is such that $\cos\theta = k$, where $0 < k < 1$.

(i) Find an expression, in terms of k, for

(a) $\sin\theta$, [2]

(b) $\tan\theta$. [1]

(ii) Explain why $\sin 2\theta$ is negative for $0 < k < 1$. [2]

J14/12/Q3

Answers: **(i) (a)** $-\sqrt{1-k^2}$ **(b)** $\frac{-\sqrt{1-k^2}}{k}$ **(ii)** Proof

27 (i) Prove the identity $\frac{1}{\cos\theta} - \frac{\cos\theta}{1+\sin\theta} \equiv \tan\theta$. [4]

(ii) Solve the equation $\frac{1}{\cos\theta} - \frac{\cos\theta}{1+\sin\theta} + 2 = 0$ for $0° \leqslant \theta \leqslant 360°$. [3]

Answers: **(i)** Proof **(ii)** 116.6°, 296.6° **J14/12/Q5**

28 (i) Prove the identity $\frac{\sin\theta - \cos\theta}{\sin\theta + \cos\theta} \equiv \frac{\tan\theta - 1}{\tan\theta + 1}$. [1]

(ii) Hence solve the equation $\frac{\sin\theta - \cos\theta}{\sin\theta + \cos\theta} = \frac{\tan\theta}{6}$, for $0° \leqslant \theta \leqslant 180°$. [4]

Answer: **(ii)** 63.4, 71.6. **J15/12/Q5**

29 A tourist attraction in a city centre is a big vertical wheel on which passengers can ride. The wheel turns in such a way that the height, h m, of a passenger above the ground is given by the formula $h = 60(1 - \cos kt)$. In this formula, k is a constant, t is the time in minutes that has elapsed since the passenger started the ride at ground level and kt is measured in radians.

(i) Find the greatest height of the passenger above the ground. [1]

One complete revolution of the wheel takes 30 minutes.

(ii) Show that $k = \frac{1}{15}\pi$. [2]

(iii) Find the time for which the passenger is above a height of 90 m. [3]

Answer: (i) 120; (iii) 10. J15/12/Q6

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30 (i) Show that the equation $2\cos x = 3\tan x$ can be written as a quadratic equation in $\sin x$. [3]

ii) Solve the equation $2\cos 2y = 3\tan 2y$, for $0° \leqslant y \leqslant 180°$. [4]

Answers: **(i)** $2\sin^2 x + 3\sin x - 2 = 0$; **(ii)** 15°, 75°. N12/12/Q6

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31 Given that $\cos x = p$, where x is an acute angle in degrees, find, in terms of p,

(i) $\sin x$, [1]

(ii) $\tan x$, [1]

(iii) $\tan(90° - x)$. [1]

Answer: **(i)** $\sqrt{1-p^2}$, **(ii)** $\frac{\sqrt{1-p^2}}{p}$, **(iii)** $\frac{p}{\sqrt{1-p^2}}$. N13/12/Q1

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32 (i) Show that the equation $1 + \sin x \tan x = 5\cos x$ can be expressed as

$$6\cos^2 x - \cos x - 1 = 0.$$ [3]

(ii) Hence solve the equation $1 + \sin x \tan x = 5\cos x$ for $0° \leqslant x \leqslant 180°$. [3]

Answer: **(ii)** 60°, 109.5° N14/12/Q5

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33 (i) Prove the identity $\left(\frac{1}{\sin x} - \frac{1}{\tan x}\right)^2 \equiv \frac{1 - \cos x}{1 + \cos x}$. [4]

(ii) Hence solve the equation $\left(\frac{1}{\sin x} - \frac{1}{\tan x}\right)^2 = \frac{2}{5}$ for $0 \leqslant x \leqslant 2\pi$. [3]

Answer: **(ii)** x = 1.13, 5.16 N15/12/Q4

Homework: Trigonometry – Variants 11 & 13

1 (i) Express the equation $3\sin\theta = \cos\theta$ in the form $\tan\theta = k$ and solve the equation for $0° < \theta < 180°$. [2]

(ii) Solve the equation $3\sin^2 2x = \cos^2 2x$ for $0° < x < 180°$. [4]

Answers: (i) $\theta = 18.4°$; (ii) 15°, 75°, 105°, 165°. 13/J15/4

2 Given that θ is an obtuse angle measured in radians and that $\sin\theta = k$, find, in terms of k, an expression for

(i) $\cos\theta$, [1]

(ii) $\tan\theta$, [2]

(iii) $\sin(\theta + \pi)$. [1]

Answer: (i) $-\sqrt{1-k^2}$ (ii) $\dfrac{-k}{\sqrt{1-k^2}}$ (iii) $-k$ 11/J15/1

3 (i) Show that $\sin^4\theta - \cos^4\theta \equiv 2\sin^2\theta - 1$. [3]

(ii) Hence solve the equation $\sin^4\theta - \cos^4\theta = \frac{1}{2}$ for $0° \leqslant \theta \leqslant 360°$. [4]

Answer: (ii) 60°, 120°, 240°, 300° 13/N14/5

4 Find the value of x satisfying the equation $\sin^{-1}(x - 1) = \tan^{-1}(3)$. [3]

Answer: $x = 1 + 3\sqrt{10}$ (or 1.95) 11/N14/2

5 Solve the equation $\dfrac{13\sin^2\theta}{2 + \cos\theta} + \cos\theta = 2$ for $0° \leqslant \theta \leqslant 180°$. [4]

Answers: $x = 30°$ and 150° 11/N14/3

6 (i) Prove the identity $\dfrac{\tan x + 1}{\sin x \tan x + \cos x} \equiv \sin x + \cos x$. [3]

(ii) Hence solve the equation $\dfrac{\tan x + 1}{\sin x \tan x + \cos x} = 3\sin x - 2\cos x$ for $0 \leqslant x \leqslant 2\pi$. [3]

Answer: (ii) 0.983, 4.12. 13/J14/4

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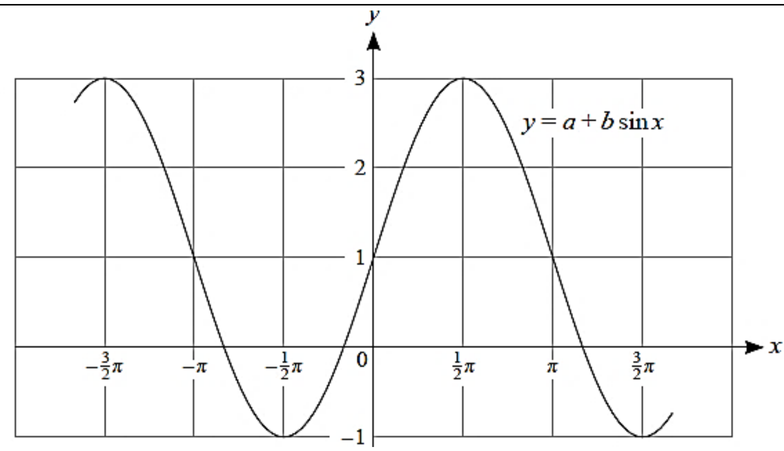

The diagram shows part of the graph of $y = a + b\sin x$. State the values of the constants a and b. [2]

Answers: $a = 1$, $b = 2$. 11/J14/1

8 (i) Prove the identity $\dfrac{\sin\theta}{1-\cos\theta} - \dfrac{1}{\sin\theta} \equiv \dfrac{1}{\tan\theta}$. [4]

(ii) Hence solve the equation $\dfrac{\sin\theta}{1-\cos\theta} - \dfrac{1}{\sin\theta} = 4\tan\theta$ for $0° < \theta < 180°$. [3]

Answer: (ii) 26.6°, 153.4°. 11/J14/9

9 (a) Find the possible values of x for which $\sin^{-1}(x^2 - 1) = \frac{1}{3}\pi$, giving your answers correct to 3 decimal places. [3]

(b) Solve the equation $\sin(2\theta + \frac{1}{3}\pi) = \frac{1}{2}$ for $0 \leqslant \theta \leqslant \pi$, giving θ in terms of π in your answers. [4]

Answer: ±1.366, $\dfrac{\pi}{4}$, $\dfrac{11\pi}{12}$ 13/N13/7

10 (i) Solve the equation $4\sin^2 x + 8\cos x - 7 = 0$ for $0° \leqslant x \leqslant 360°$. [4]

(ii) Hence find the solution of the equation $4\sin^2\left(\frac{1}{2}\theta\right) + 8\cos\left(\frac{1}{2}\theta\right) - 7 = 0$ for $0° \leqslant \theta \leqslant 360°$. [2]

Answers: (i) 60º, 300º; (ii) 120º. 11/N13/4

11 (i) Express the equation $2\cos^2\theta = \tan^2\theta$ as a quadratic equation in $\cos^2\theta$. [2]

(ii) Solve the equation $2\cos^2\theta = \tan^2\theta$ for $0 \leqslant \theta \leqslant \pi$, giving solutions in terms of π. [3]

(ii) ¼π, ¾ π. 13/J13/3

12 (i) Sketch, on the same diagram, the curves $y = \sin 2x$ and $y = \cos x - 1$ for $0 \leqslant x \leqslant 2\pi$. [4]

(ii) Hence state the number of solutions, in the interval $0 \leqslant x \leqslant 2\pi$, of the equations

(a) $2\sin 2x + 1 = 0$, [1]

(b) $\sin 2x - \cos x + 1 = 0$. [1]

13/J13/5

Answer: **(i)** Sketch; **(ii): (a)** 4 solutions, **(b)** 3 solutions.

13 (i) Show that $\dfrac{\sin\theta}{\sin\theta + \cos\theta} + \dfrac{\cos\theta}{\sin\theta - \cos\theta} \equiv \dfrac{1}{\sin^2\theta - \cos^2\theta}$. [3]

(ii) Hence solve the equation $\dfrac{\sin\theta}{\sin\theta + \cos\theta} + \dfrac{\cos\theta}{\sin\theta - \cos\theta} = 3$, for $0° \leqslant \theta \leqslant 360°$. [4]

11/J13/5

Answers: **(ii)** 54.7°, 125.3°, 234.7°, 305.3°.

14 Solve the equation $7\cos x + 5 = 2\sin^2 x$, for $0° \leqslant x \leqslant 360°$. [4]

13/N12/3

Answers: 120°, 240°.

15 (i) Solve the equation $2\cos^2\theta = 3\sin\theta$, for $0° \leqslant \theta \leqslant 360°$. [4]

(ii) The smallest positive solution of the equation $2\cos^2(n\theta) = 3\sin(n\theta)$, where n is a positive integer, is 10°. State the value of n and hence find the largest solution of this equation in the interval $0° \leqslant \theta \leqslant 360°$. [3]

11/N12/7

Answers: **(i)** 30°, 150°; **(ii)** 290°.

16 (i) Prove the identity $\tan^2\theta - \sin^2\theta \equiv \tan^2\theta \sin^2\theta$. [3]

(ii) Use this result to explain why $\tan\theta > \sin\theta$ for $0° < \theta < 90°$. [1]

13/J12/1

Answers: **(i)** Proof **(ii)** Explanation

17 (i) Solve the equation $\sin 2x + 3\cos 2x = 0$ for $0° \leqslant x \leqslant 360°$. [5]

(ii) How many solutions has the equation $\sin 2x + 3\cos 2x = 0$ for $0° \leqslant x \leqslant 1080°$? [1]

13/J12/4

Answers: **(i)** x = 54.2° or 144.2° or 234.2° or 324.2° **(ii)** 12 solutions

18 Solve the equation $\sin 2x = 2\cos 2x$, for $0° \leqslant x \leqslant 180°$. [4]

Answers: 31.7° 122° 11/J12/1

19 **(i)** Given that

$$3\sin^2 x - 8\cos x - 7 = 0,$$

show that, for real values of x,

$$\cos x = -\tfrac{2}{3}.$$ [3]

(ii) Hence solve the equation

$$3\sin^2(\theta + 70°) - 8\cos(\theta + 70°) - 7 = 0$$

for $0° \leqslant \theta \leqslant 180°$. [4]

Answers: **(ii)** 61.8, 158.2 13/N11/5

20 **(i)** Sketch, on a single diagram, the graphs of $y = \cos 2\theta$ and $y = \frac{1}{2}$ for $0 \leqslant \theta \leqslant 2\pi$. [3]

(ii) Write down the number of roots of the equation $2\cos 2\theta - 1 = 0$ in the interval $0 \leqslant \theta \leqslant 2\pi$. [1]

(iii) Deduce the number of roots of the equation $2\cos 2\theta - 1 = 0$ in the interval $10\pi \leqslant \theta \leqslant 20\pi$. [1]

Answers: **(ii)** 4; **(iii)** 20. 11/N11/3

21 **(i)** Prove the identity $\left(\frac{1}{\sin\theta} - \frac{1}{\tan\theta}\right)^2 \equiv \frac{1-\cos\theta}{1+\cos\theta}$. [3]

(ii) Hence solve the equation $\left(\frac{1}{\sin\theta} - \frac{1}{\tan\theta}\right)^2 = \frac{2}{5}$, for $0° \leqslant \theta \leqslant 360°$. [4]

Answer: **(ii)** 64.6°, 295.4°. 13/J11/8

22 **(i)** Show that the equation $2\tan^2\theta\sin^2\theta = 1$ can be written in the form

$$2\sin^4\theta + \sin^2\theta - 1 = 0.$$ [2]

(ii) Hence solve the equation $2\tan^2\theta\sin^2\theta = 1$ for $0° \leqslant \theta \leqslant 360°$. [4]

Answer: **(ii)** 45°, 135°, 225°, 315°. 11/J11/5

23 Solve the equation $15\sin^2 x = 13 + \cos x$ for $0° \leqslant x \leqslant 180°$. [4]

Answer: 70.5°, 113.6°. 13/N10/3

24 (i) Sketch the curve $y = 2\sin x$ for $0 \leqslant x \leqslant 2\pi$. [1]

(ii) By adding a suitable straight line to your sketch, determine the number of real roots of the equation

$$2\pi \sin x = \pi - x.$$

State the equation of the straight line. [3]

13/N10/4

Answers: (ii) 3, $y = 1 - \frac{x}{\pi}$.

25 In the expansion of $(1 + ax)^6$, where a is a constant, the coefficient of x is -30. Find the coefficient of x^3. [4]

11/N10/2

Answer. -2500.

26 (i) Prove the identity $\frac{\sin x \tan x}{1 - \cos x} \equiv 1 + \frac{1}{\cos x}$. [3]

(ii) Hence solve the equation $\frac{\sin x \tan x}{1 - \cos x} + 2 = 0$, for $0° \leqslant x \leqslant 360°$. [3]

11/N10/4

Answer. (ii) 109.5°, 250.5°.

27 (i) Show that the equation $2\sin x \tan x + 3 = 0$ can be expressed as $2\cos^2 x - 3\cos x - 2 = 0$. [2]

(ii) Solve the equation $2\sin x \tan x + 3 = 0$ for $0° \leqslant x \leqslant 360°$. [3]

13/J10/4

Answer. (ii) 120°, 240°.

28 The acute angle x radians is such that $\tan x = k$, where k is a positive constant. Express, in terms of k,

(i) $\tan(\pi - x)$, [1]

(ii) $\tan(\frac{1}{2}\pi - x)$, [1]

(iii) $\sin x$. [2]

11/J10/1

Answers: (i) $-k$; (ii) $\frac{1}{k}$; (iii) $\frac{k}{\sqrt{1+k^2}}$.

29 Solve the equation $\sin^{-1}(4x^4 + x^2) = \frac{1}{6}\pi$. [4]

N15/11/Q3

(i) Show that the equation $\frac{4\cos\theta}{\tan\theta} + 15 = 0$ can be expressed as

$$4\sin^2\theta - 15\sin\theta - 4 = 0. \quad [3]$$

(ii) Hence solve the equation $\frac{4\cos\theta}{\tan\theta} + 15 = 0$ for $0° \leqslant \theta \leqslant 360°$. [3]

N15/11/Q4

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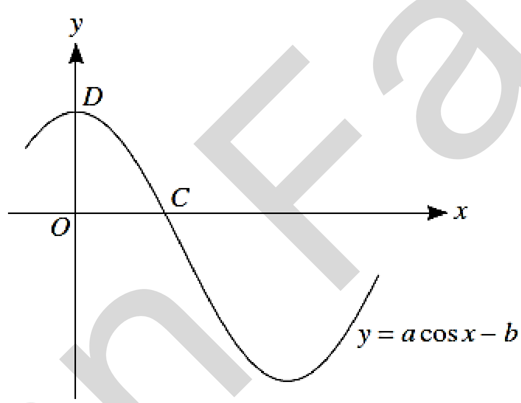
30 (a) Show that the equation $\frac{1}{\cos\theta} + 3\sin\theta\tan\theta + 4 = 0$ can be expressed as

$$3\cos^2\theta - 4\cos\theta - 4 = 0,$$

and hence solve the equation $\frac{1}{\cos\theta} + 3\sin\theta\tan\theta + 4 = 0$ for $0° \leqslant \theta \leqslant 360°$. [6]

(b)



The diagram shows part of the graph of $y = a\cos x - b$, where a and b are constants. The graph crosses the x-axis at the point $C(\cos^{-1}c, 0)$ and the y-axis at the point $D(0, d)$. Find c and d in terms of a and b. [2]

N15/13/Q7

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31 Solve the equation $3\sin^2\theta = 4\cos\theta - 1$ for $0° \leqslant \theta \leqslant 360°$. [4]

Answer: $\theta = 48.2°, 311.8°$ **J16/11/Q2**

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32 (i) Show that $3\sin x\tan x - \cos x + 1 = 0$ can be written as a quadratic equation in $\cos x$ and hence solve the equation $3\sin x\tan x - \cos x + 1 = 0$ for $0 \leqslant x \leqslant \pi$. [5]

(ii) Find the solutions to the equation $3\sin 2x\tan 2x - \cos 2x + 1 = 0$ for $0 \leqslant x \leqslant \pi$. [3]

Answers: (i) 2.42, 0; (ii) 1.21, 1.93, 0, 3.14. **J16/13/Q8**

33 (i) Prove the identity $\frac{1+\cos\theta}{\sin\theta}+\frac{\sin\theta}{1+\cos\theta}\equiv\frac{2}{\sin\theta}$. [3]

(ii) Hence solve the equation $\frac{1+\cos\theta}{\sin\theta}+\frac{\sin\theta}{1+\cos\theta}=\frac{3}{\cos\theta}$ for $0° \leqslant \theta \leqslant 360°$. [3]

Answers: **(ii)** 33.7° and 213.7° **J17/11/Q3**

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34 The equation of a curve is $y = 2\cos x$.

(i) Sketch the graph of $y = 2\cos x$ for $-\pi \leqslant x \leqslant \pi$, stating the coordinates of the point of intersection with the y-axis. [2]

Points P and Q lie on the curve and have x-coordinates of $\frac{1}{3}\pi$ and π respectively.

(ii) Find the length of PQ correct to 1 decimal place. [2]

9

The line through P and Q meets the x-axis at $H(h, 0)$ and the y-axis at $K(0, k)$.

(iii) Show that $h = \frac{5}{9}\pi$ and find the value of k. [3]

Answer: **(ii)** 3.7 **(iii)** 2.5 **J17/11/Q5**

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35 (i) Show that the equation $\frac{2\sin\theta+\cos\theta}{\sin\theta+\cos\theta}=2\tan\theta$ may be expressed as $\cos^2\theta = 2\sin^2\theta$. [3]

(ii) Hence solve the equation $\frac{2\sin\theta+\cos\theta}{\sin\theta+\cos\theta}=2\tan\theta$ for $0° < \theta < 180°$. [3]

Answer : **(ii)** 35.3°, 144.7°. **J17/13/Q5**

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36 (i) Prove the identity $(\sin\theta+\cos\theta)(1-\sin\theta\cos\theta)\equiv\sin^3\theta+\cos^3\theta$. [3]

7

(ii) Hence solve the equation $(\sin\theta+\cos\theta)(1-\sin\theta\cos\theta)=3\cos^3\theta$ for $0° \leqslant \theta \leqslant 360°$. [3]

Answer: **(ii)** 51.6° and 231.6° **J18/11/Q4**

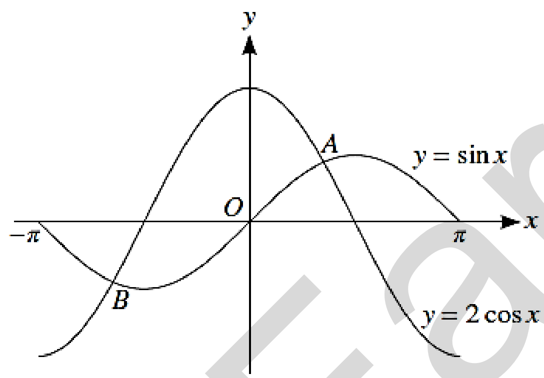
37 (a) (i) Express $\dfrac{\tan^2\theta - 1}{\tan^2\theta + 1}$ in the form $a\sin^2\theta + b$, where a and b are constants to be found. [3]

(ii) Hence, or otherwise, and showing all necessary working, solve the equation

$$\frac{\tan^2\theta - 1}{\tan^2\theta + 1} = \frac{1}{4}$$

for $-90° \leqslant \theta \leqslant 0°$. [2]

(b)



The diagram shows the graphs of $y = \sin x$ and $y = 2\cos x$ for $-\pi \leqslant x \leqslant \pi$. The graphs intersect at the points A and B.

(i) Find the x-coordinate of A. [2]

(ii) Find the y-coordinate of B. [2]

Answers: **(a)(i)** $2\sin^2\theta - 1$; **(a)(ii)** $-52.2°$; **(b)(i)** 1.11; **(b)(ii)** -0.894 or -0.895 or -0.896. **J18/13/Q7**

38 (i) Show that $\cos^4 x \equiv 1 - 2\sin^2 x + \sin^4 x$. [1]

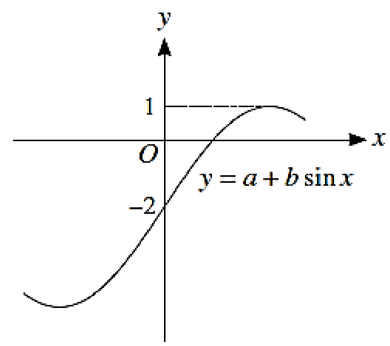
(ii) Hence, or otherwise, solve the equation $8\sin^4 x + \cos^4 x = 2\cos^2 x$ for $0° \leqslant x \leqslant 360°$. [5]

Answers: **(ii)** 35.3°, 144.7°, 215.3°, 324.7° **N16/11/Q6**

39 Showing all necessary working, solve the equation $6\sin^2 x - 5\cos^2 x = 2\sin^2 x + \cos^2 x$ for $0° \leqslant x \leqslant 360°$. [4]

N16/13/Q3

40 **(a)**



The diagram shows part of the graph of $y = a + b\sin x$. Find the values of the constants a and b. [2]

(b) **(i)** Show that the equation

$$(\sin\theta + 2\cos\theta)(1 + \sin\theta - \cos\theta) = \sin\theta(1 + \cos\theta)$$

may be expressed as $3\cos^2\theta - 2\cos\theta - 1 = 0$. [3]

(ii) Hence solve the equation

$$(\sin\theta + 2\cos\theta)(1 + \sin\theta - \cos\theta) = \sin\theta(1 + \cos\theta)$$

for $-180° \leqslant \theta \leqslant 180°$. [4]

Answers: **(a)** $a = -2$, $b = 3$ **(b)(ii)** 0°, ± 109.5° **N17/11/Q7**

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41 (i) Show that the equation $\dfrac{\cos\theta + 4}{\sin\theta + 1} + 5\sin\theta - 5 = 0$ may be expressed as $5\cos^2\theta - \cos\theta - 4 = 0$. [3]

(ii) Hence solve the equation $\dfrac{\cos\theta + 4}{\sin\theta + 1} + 5\sin\theta - 5 = 0$ for $0° \leqslant \theta \leqslant 360°$. [4]

Answer: **(ii)** 0°, 360°, 143.1°, 216.9° **N17/13/Q5**

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42 **(i)** Show that the equation

$$\frac{\cos\theta - 4}{\sin\theta} - \frac{4\sin\theta}{5\cos\theta - 2} = 0$$

may be expressed as $9\cos^2\theta - 22\cos\theta + 4 = 0$. [3]

(ii) Hence solve the equation

$$\frac{\cos\theta - 4}{\sin\theta} - \frac{4\sin\theta}{5\cos\theta - 2} = 0$$

for $0° \leqslant \theta \leqslant 360°$. [3]

Answers: **(ii)** 78.6° and 218.4° N18/11/Q5

43

(i) Show that $\dfrac{\tan\theta + 1}{1 + \cos\theta} + \dfrac{\tan\theta - 1}{1 - \cos\theta} \equiv \dfrac{2(\tan\theta - \cos\theta)}{\sin^2\theta}$. [3]

(ii) Hence, showing all necessary working, solve the equation

$$\frac{\tan\theta + 1}{1 + \cos\theta} + \frac{\tan\theta - 1}{1 - \cos\theta} = 0$$

for $0° < \theta < 90°$. [4]

Answer: **(ii)** 38.2° N18/13/Q7

Circular Measure

Example 1

(i) Express in radians (a) 30° (b) 315° (c) 29°.

(ii) Express in degrees (a) $\frac{\pi}{12}$ (b) $\frac{8\pi}{3}$ (c) 1.2 radians.

Arc length, $s = r\theta$ Sector area, $A = \frac{1}{2}r^2\theta$

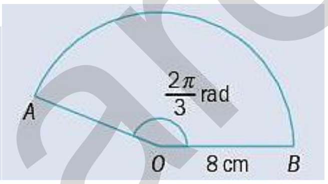
where θ is measured in radians and r is the radius of the circle.

Example 2

OAB is a sector of a circle with centre *O* and radius 8 cm.

a) Find the area of the sector in terms of π.

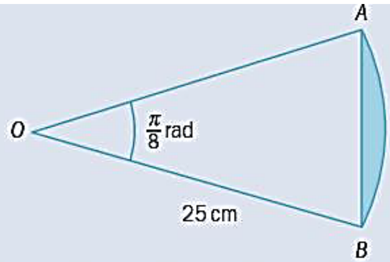
b) Find the perimeter of the sector in terms of π.



Example 3

OAB is a sector of a circle with centre *O* and radius 25 cm.
The line *AB* is a **chord** of the circle.

Find the area of the shaded region.
(You are given that the area of triangle *OAB* may be found by using the formula $\frac{1}{2} \times OA \times OB \times \sin AOB$.)



Pythagoras' theorem $a^2 + b^2 = c^2$

Right-angled triangle trigonometry $\sin x = \frac{O}{H}, \cos x = \frac{A}{H}, \tan x = \frac{O}{A}$

Sine rule $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

Cosine rule $a^2 = b^2 + c^2 - 2bc\cos A$

Area of a triangle rule $\text{Area} = \frac{1}{2}ab\sin C$

Example 4

In the diagram, BD is the arc of a circle with centre C and radius 20 m.

Angle $BCD = \frac{2\pi}{3}$ radians.

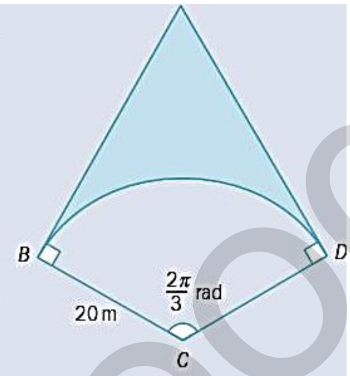
AB and AD are tangents to the circle at B and D, respectively.

a) Show that $AB = 20\sqrt{3}$ cm.

b) Show that the area of the shaded region is $400\left(\sqrt{3} - \frac{\pi}{3}\right)$ cm².

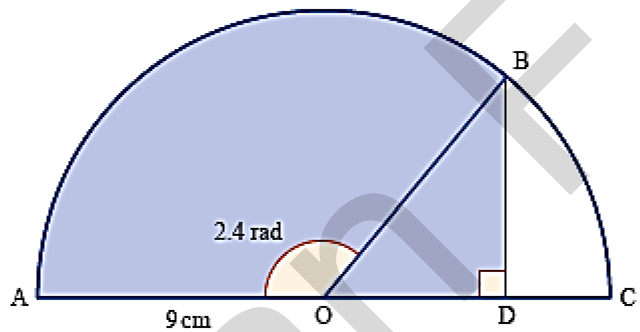
c) Find the perimeter of the shaded region in terms of π and $\sqrt{3}$.

Note: You may assume that $\tan \frac{\pi}{3} = \sqrt{3}$ in this question. Exact trigonometric ratios will be studied in Chapter 5.



Example 5

In the diagram, ABC is a semi-circle, centre O and radius 9 cm. The line BD is perpendicular to the diameter AC and angle AOB = 2.4 radians.



(I) Show that BD = 6.08 cm, correct to 3 significant figures.

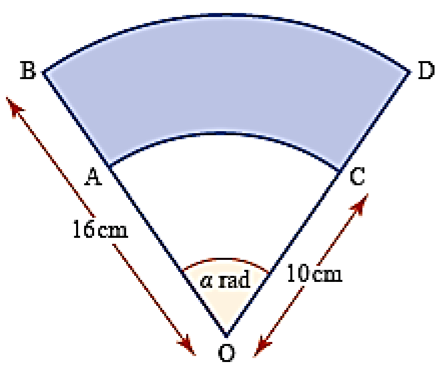
(II) Find the perimeter of the shaded region.

(III) Find the area of the shaded region.

[Cambridge AS & A Level Mathematics 9709, Paper 1 Q8 June 2005]

Example 6

In the diagram, OAB and OCD are radii of a circle, centre O and radius 16 cm. Angle AOC = α radians. AC and BD are arcs of circles, centre O and radii 10 cm and 16 cm respectively.



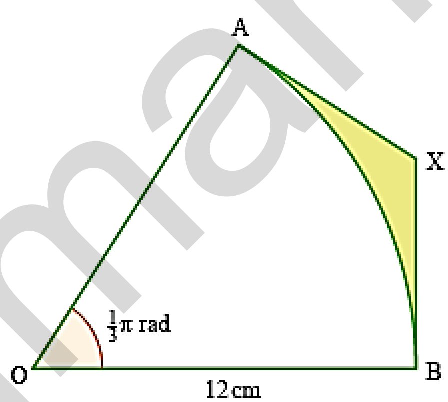
(i) In the case where $\alpha = 0.8$, find the area of the shaded region.

(ii) Find the value of α for which the perimeter of the shaded region is 28.9 cm.

[Cambridge AS & A Level Mathematics 9709, Paper 1 Q2 November 2005]

Example 7

In the diagram, OAB is a sector of a circle with centre O and radius 12 cm. The lines AX and BX are tangents to the circle at A and B respectively. Angle AOB = $\frac{1}{3}\pi$ radians.



(i) Find the exact length of AX, giving your answer in terms of $\sqrt{3}$.

(ii) Find the area of the shaded region, giving your answer in terms of π and $\sqrt{3}$.

[Cambridge AS & A Level Mathematics 9709, Paper 1 Q5 June 2007]

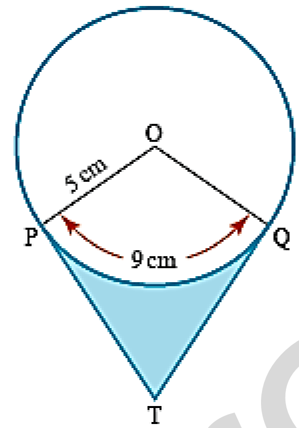
Example 8

In the diagram, the circle has centre O and radius 5 cm. The points P and Q lie on the circle, and the arc length PQ is 9 cm. The tangents to the circle at P and Q meet at the point T. Calculate

(i) angle POQ in radians

(ii) the length of PT

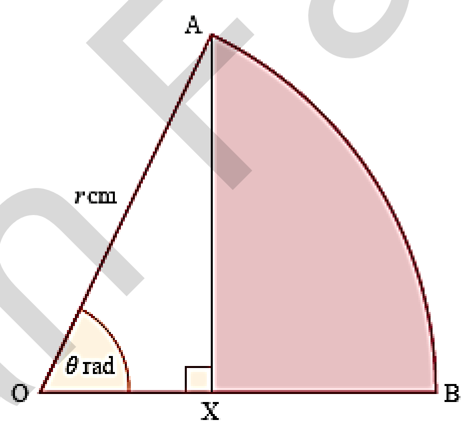
(iii) the area of the shaded region.



[Cambridge AS & A Level Mathematics 9709, Paper 1 Q6 November 2008]

Example 9

In the diagram, AB is an arc of a circle, centre O and radius r cm, and angle AOB = θ radians. The point X lies on OB and AX is perpendicular to OB.



I) Show that the area, A cm^2, of the shaded region AXB is given by

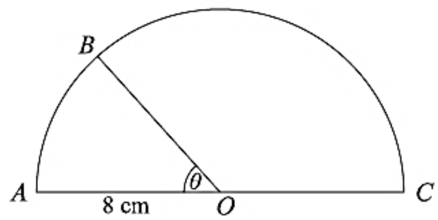
$$A = \frac{1}{2}r^2(\theta - \sin\theta\cos\theta)$$

II) In the case where $r = 12$ and $\theta = \frac{1}{6}\pi$, find the perimeter of the shaded region AXB, leaving your answer in terms of $\sqrt{3}$ and π.

[Cambridge AS & A Level Mathematics 9709, Paper 1 Q7 November 2007]

Homework: Circular Measure Variant 12

1



The diagram shows a semicircle ABC with centre O and radius 8 cm. Angle $AOB = \theta$ radians.

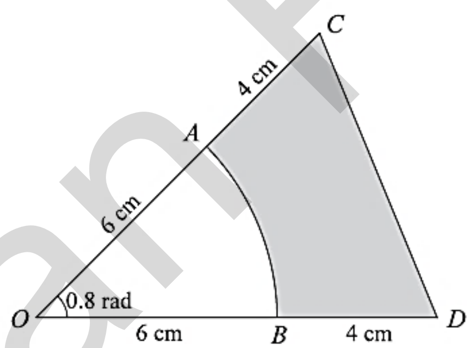
(i) In the case where $\theta = 1$, calculate the area of the sector BOC. [3]

(ii) Find the value of θ for which the perimeter of sector AOB is one half of the perimeter of sector BOC. [3]

(iii) In the case where $\theta = \frac{1}{3}\pi$, show that the exact length of the perimeter of triangle ABC is $(24 + 8\sqrt{3})$ cm. [3]

Answers: (i) 68.5 cm²; (ii) 0.381; (iii) Proof. J03/Q9

2



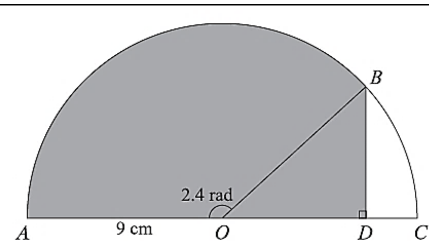
In the diagram, OCD is an isosceles triangle with $OC = OD = 10$ cm and angle $COD = 0.8$ radians. The points A and B, on OC and OD respectively, are joined by an arc of a circle with centre O and radius 6 cm. Find

(i) the area of the shaded region, [3]

(ii) the perimeter of the shaded region. [4]

Answers: (i) 21.5 cm^2; (ii) 20.6 cm. J04/Q5

3



In the diagram, ABC is a semicircle, centre O and radius 9 cm. The line BD is perpendicular to the diameter AC and angle $AOB = 2.4$ radians.

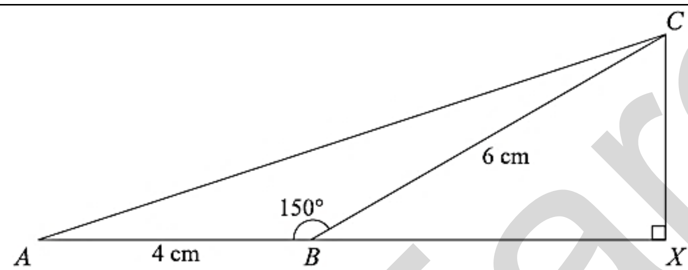
(i) Show that $BD = 6.08$ cm, correct to 3 significant figures. [2]

(ii) Find the perimeter of the shaded region. [3]

(iii) Find the area of the shaded region. [3]

Answers: (ii) 43.3 cm; (iii) 117 cm^2. **J05/Q8**

4



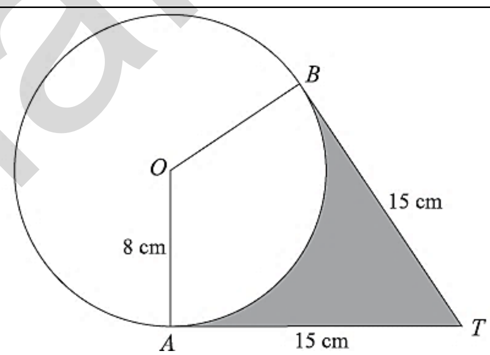
In the diagram, ABC is a triangle in which $AB = 4$ cm, $BC = 6$ cm and angle $ABC = 150°$. The line CX is perpendicular to the line ABX.

(i) Find the exact length of BX and show that angle $CAB = \tan^{-1}\left(\frac{3}{4+3\sqrt{3}}\right)$. [4]

(ii) Show that the exact length of AC is $\sqrt{(52+24\sqrt{3})}$ cm. [2]

Answer: (i) $3\sqrt{3}$. **J06/Q6**

5



The diagram shows a circle with centre O and radius 8 cm. Points A and B lie on the circle. The tangents at A and B meet at the point T, and $AT = BT = 15$ cm.

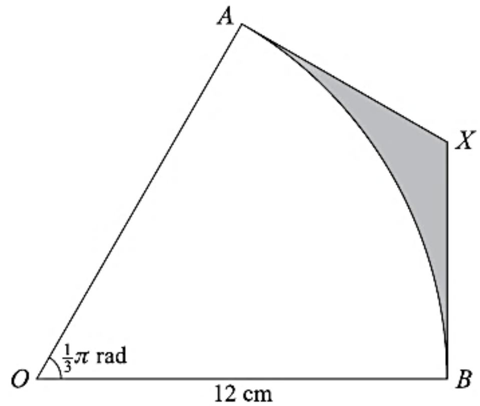
(i) Show that angle AOB is 2.16 radians, correct to 3 significant figures. [3]

(ii) Find the perimeter of the shaded region. [2]

(iii) Find the area of the shaded region. [3]

Answers: (ii) 47.3 cm; (iii) 50.9 (±0.1) cm². J06/Q7

6



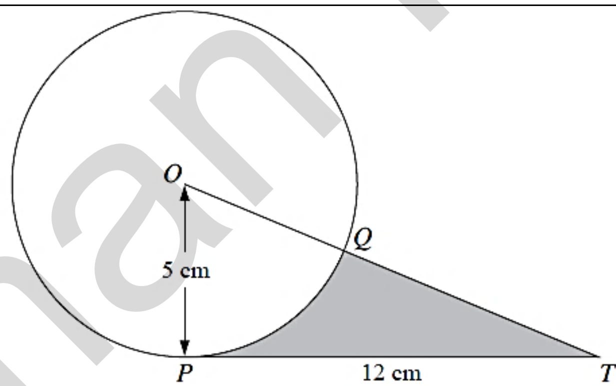
In the diagram, OAB is a sector of a circle with centre O and radius 12 cm. The lines AX and BX are tangents to the circle at A and B respectively. Angle $AOB = \frac{1}{3}\pi$ radians.

(i) Find the exact length of AX, giving your answer in terms of $\sqrt{3}$. [2]

(ii) Find the area of the shaded region, giving your answer in terms of π and $\sqrt{3}$. [3]

Answers: (i) $4\sqrt{3}$; (ii) $48\sqrt{3} - 24\pi$. J07/Q5

7



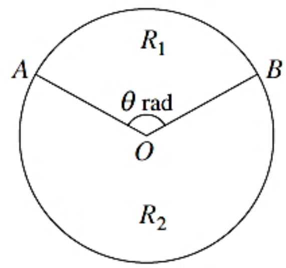
The diagram shows a circle with centre O and radius 5 cm. The point P lies on the circle, PT is a tangent to the circle and $PT = 12$ cm. The line OT cuts the circle at the point Q.

(i) Find the perimeter of the shaded region. [4]

(ii) Find the area of the shaded region. [3]

Answers: (i) 25.9 cm; (ii) 15.3 cm². J08/Q5

8



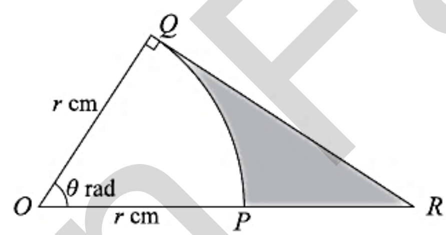
The diagram shows a circle with centre O. The circle is divided into two regions, R_1 and R_2, by the radii OA and OB, where angle $AOB = \theta$ radians. The perimeter of the region R_1 is equal to the length of the major arc AB.

(i) Show that $\theta = \pi - 1$. [3]

(ii) Given that the area of region R_1 is $30\,\text{cm}^2$, find the area of region R_2, correct to 3 significant figures. [4]

Answers: (i) Proof; **(ii)** 58.0° or 57.9° J09/Q5

9



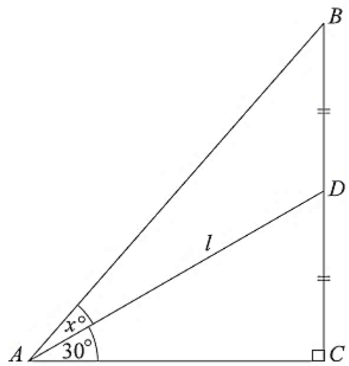
In the diagram, OPQ is a sector of a circle, centre O and radius r cm. Angle $QOP = \theta$ radians. The tangent to the circle at Q meets OP extended at R.

(i) Show that the area, $A\,\text{cm}^2$, of the shaded region is given by $A = \frac{1}{2}r^2(\tan\theta - \theta)$. [2]

(ii) In the case where $\theta = 0.8$ and $r = 15$, evaluate the length of the perimeter of the shaded region. [4]

Answers: (i) Proof; (ii) 34.0 cm. N02/Q3

10



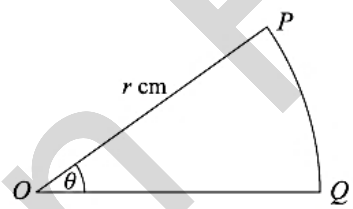
In the diagram, triangle ABC is right-angled and D is the mid-point of BC. Angle $DAC = 30°$ and angle $BAD = x°$. Denoting the length of AD by l,

(i) express each of AC and BC exactly in terms of l, and show that $AB = \frac{1}{2}l\sqrt{7}$, [4]

(ii) show that $x = \tan^{-1}\left(\frac{2}{\sqrt{3}}\right) - 30$. [2]

Answers: (i) $AC = \frac{1}{2}l\sqrt{3}$, $BC = l$, Proof; (ii) Proof. **N02/Q6**

11



The diagram shows the sector OPQ of a circle with centre O and radius r cm. The angle POQ is θ radians and the perimeter of the sector is 20 cm.

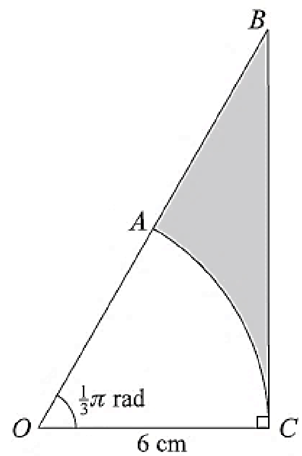
(i) Show that $\theta = \frac{20}{r} - 2$. [2]

(ii) Hence express the area of the sector in terms of r. [2]

(iii) In the case where $r = 8$, find the length of the chord PQ. [3]

Answers: (ii) $A = 10r - r^2$; (iii) 3.96 cm. **N03/Q6**

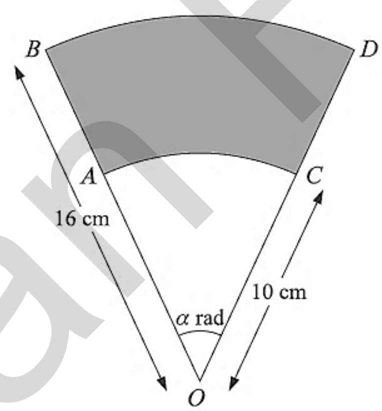
12



In the diagram, AC is an arc of a circle, centre O and radius 6 cm. The line BC is perpendicular to OC and OAB is a straight line. Angle $AOC = \frac{1}{3}\pi$ radians. Find the area of the shaded region, giving your answer in terms of π and $\sqrt{3}$. [5]

Answer: $18\sqrt{3} - 6\pi$ cm^2. **N04/Q3**

13



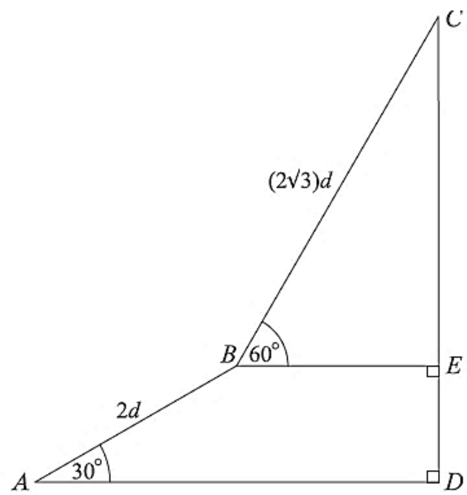
In the diagram, OAB and OCD are radii of a circle, centre O and radius 16 cm. Angle $AOC = \alpha$ radians. AC and BD are arcs of circles, centre O and radii 10 cm and 16 cm respectively.

(i) In the case where $\alpha = 0.8$, find the area of the shaded region. [2]

(ii) Find the value of α for which the perimeter of the shaded region is 28.9 cm. [3]

Answers: (i) 62.4 cm²; (ii) 0.65. **N05/Q2**

14

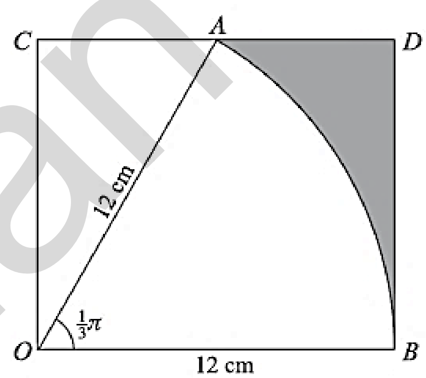


In the diagram, $ABED$ is a trapezium with right angles at E and D, and CED is a straight line. The lengths of AB and BC are $2d$ and $(2\sqrt{3})d$ respectively, and angles BAD and CBE are $30°$ and $60°$ respectively.

(i) Find the length of CD in terms of d. [2]

(ii) Show that angle $CAD = \tan^{-1}\left(\frac{2}{\sqrt{3}}\right)$. [3]

15

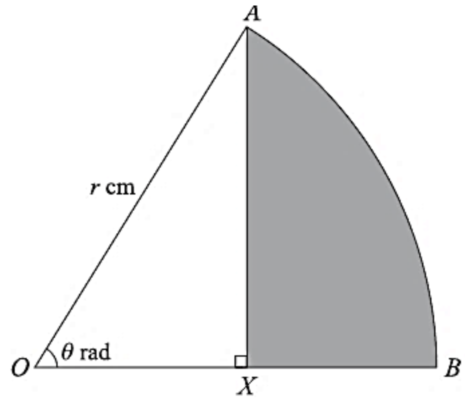


In the diagram, AOB is a sector of a circle with centre O and radius 12 cm. The point A lies on the side CD of the rectangle $OCDB$. Angle $AOB = \frac{1}{3}\pi$ radians. Express the area of the shaded region in the form $a(\sqrt{3}) - b\pi$, stating the values of the integers a and b. [6]

Answer: $a = 54$, $b = 24$.

N06/Q3

16



In the diagram, AB is an arc of a circle, centre O and radius r cm, and angle $AOB = \theta$ radians. The point X lies on OB and AX is perpendicular to OB.

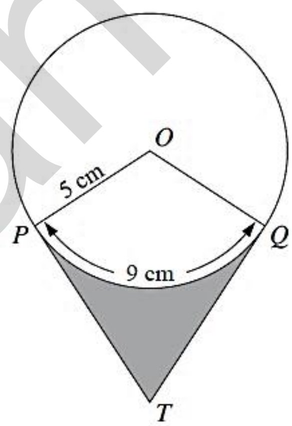
(i) Show that the area, A cm^2, of the shaded region AXB is given by

$$A = \frac{1}{2}r^2(\theta - \sin\theta\cos\theta).$$ [3]

(ii) In the case where $r = 12$ and $\theta = \frac{1}{6}\pi$, find the perimeter of the shaded region AXB, leaving your answer in terms of $\sqrt{3}$ and π. [4]

Answer: (ii) $18 - 6\sqrt{3} + 2\pi$. N07/Q7

17



In the diagram, the circle has centre O and radius 5 cm. The points P and Q lie on the circle, and the arc length PQ is 9 cm. The tangents to the circle at P and Q meet at the point T. Calculate

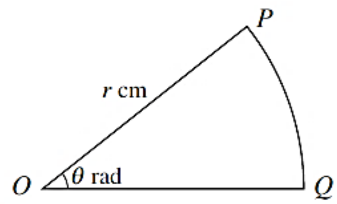
(i) angle POQ in radians, [2]

(ii) the length of PT, [3]

(iii) the area of the shaded region. [3]

Answers: (i) 1.8 radians; (ii) 6.30 cm; (iii) 9.00 cm². N08/Q6

18



A piece of wire of length 50 cm is bent to form the perimeter of a sector POQ of a circle. The radius of the circle is r cm and the angle POQ is θ radians (see diagram).

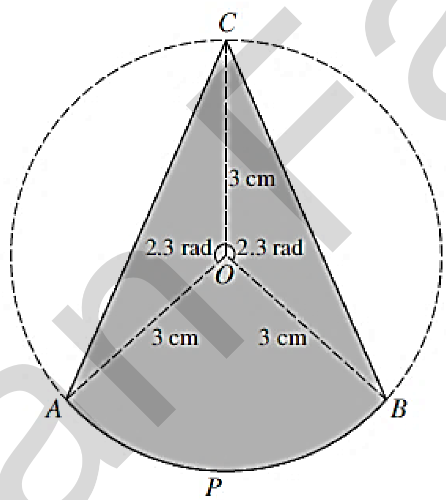
(i) Express θ in terms of r and show that the area, A cm^2, of the sector is given by

$$A = 25r - r^2.$$ [4]

(ii) Given that r can vary, find the stationary value of A and determine its nature. [4]

Answers: (ii) $156\frac{1}{4}$, Maximum. N09/12/Q7

19



The diagram shows points A, C, B, P on the circumference of a circle with centre O and radius 3 cm. Angle AOC = angle BOC = 2.3 radians.

(i) Find angle AOB in radians, correct to 4 significant figures. [1]

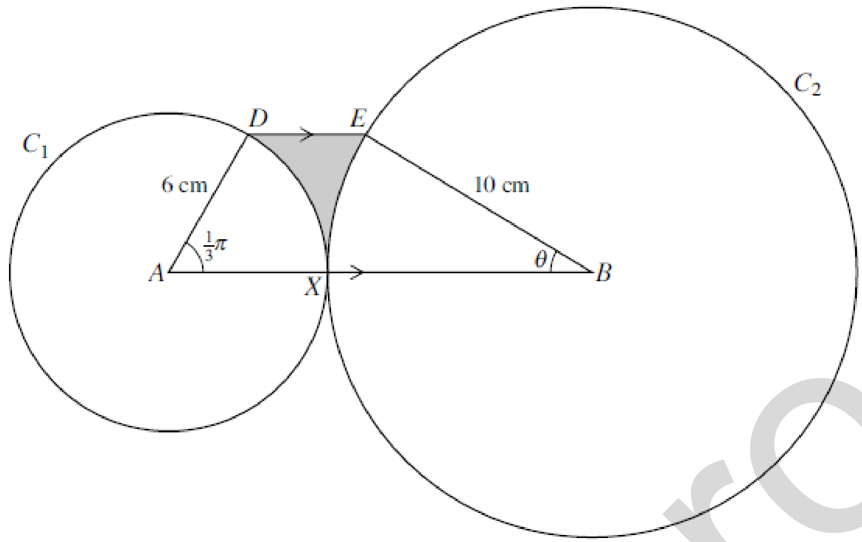
(ii) Find the area of the shaded region $ACBP$, correct to 3 significant figures. [4]

Answers: (i) 1.683; (ii) 14.3 cm^2. N10/12/Q4

20 In the triangle ABC, AB = 12 cm, angle BAC = 60° and angle ACB = 45°. Find the exact length of BC. [3]

Answer: $6\sqrt{6}$ or equivalent surd form. J08/Q1

21



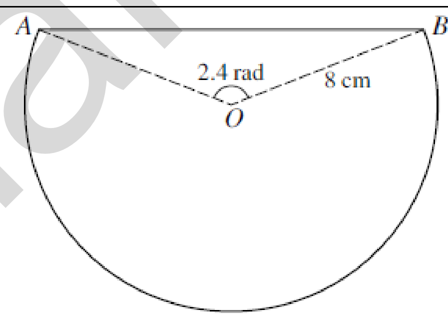
The diagram shows a circle C_1 touching a circle C_2 at a point X. Circle C_1 has centre A and radius 6 cm, and circle C_2 has centre B and radius 10 cm. Points D and E lie on C_1 and C_2 respectively and DE is parallel to AB. Angle $DAX = \frac{1}{3}\pi$ radians and angle $EBX = \theta$ radians.

(i) By considering the perpendicular distances of D and E from AB, show that the exact value of θ is $\sin^{-1}\left(\frac{3\sqrt{3}}{10}\right)$. [3]

(ii) Find the perimeter of the shaded region, correct to 4 significant figures. [5]

Answers: **(i)** Proof; **(ii)** 16.20 **N11/12/Q6**

22



The diagram shows a metal plate made by removing a segment from a circle with centre O and radius 8 cm. The line AB is a chord of the circle and angle AOB = 2.4 radians. Find

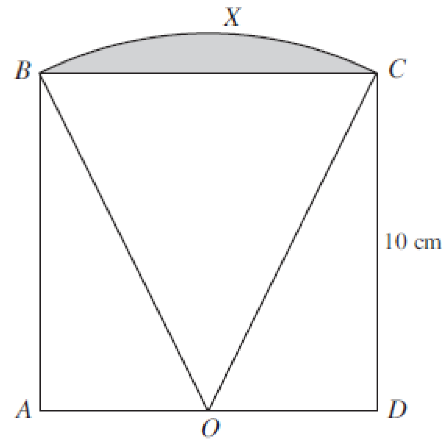
(i) the length of AB, [2]

(ii) the perimeter of the plate, [3]

(iii) the area of the plate. [3]

Answers: **(i)** 14.9 cm **(ii)** 46.0 cm **(iii)** 146 cm² **J12/12/Q6**

23



The diagram shows a square $ABCD$ of side 10 cm. The mid-point of AD is O and BXC is an arc of a circle with centre O.

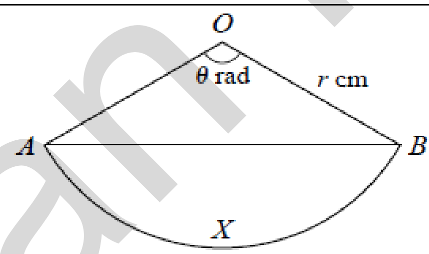
(i) Show that angle BOC is 0.9273 radians, correct to 4 decimal places. [2]

(ii) Find the perimeter of the shaded region. [3]

(iii) Find the area of the shaded region. [2]

Answers: **(i)** Proof. **(ii)** 20.4 cm. **(iii)** 7.95 or 7.96 cm². **J13/12/Q4**

24



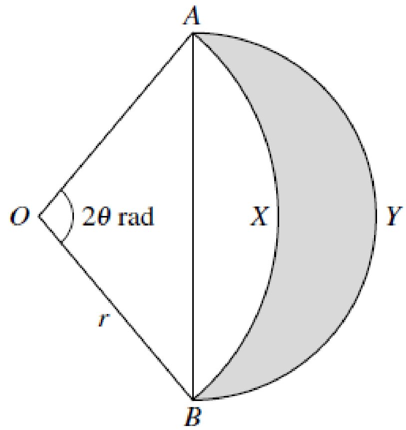
The diagram shows a sector of a circle with radius r cm and centre O. The chord AB divides the sector into a triangle AOB and a segment AXB. Angle AOB is θ radians.

(i) In the case where the areas of the triangle AOB and the segment AXB are equal, find the value of the constant p for which $\theta = p \sin \theta$. [2]

(ii) In the case where $r = 8$ and $\theta = 2.4$, find the perimeter of the segment AXB. [3]

Answers: **(i)** $p = 2$ **(ii)** 34.1 cm **J14/12/Q4**

25

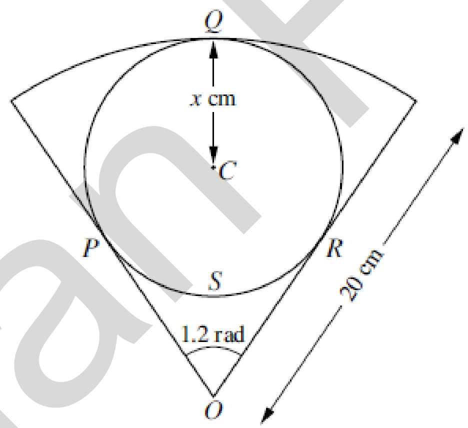


In the diagram, AYB is a semicircle with AB as diameter and $OAXB$ is a sector of a circle with centre O and radius r. Angle $AOB = 2\theta$ radians. Find an expression, in terms of r and θ, for the area of the shaded region. [4]

Answer: $\frac{1}{2}\pi r^2 \sin^2\theta - r^2\theta + \frac{1}{2}r^2\sin^2 2\theta$

J15/12/Q2

26



The diagram shows a sector of a circle with centre O and radius 20 cm. A circle with centre C and radius x cm lies within the sector and touches it at P, Q and R. Angle $POR = 1.2$ radians.

(i) Show that $x = 7.218$, correct to 3 decimal places. [4]

(ii) Find the total area of the three parts of the sector lying outside the circle with centre C. [2]

(iii) Find the perimeter of the region $OPSR$ bounded by the arc PSR and the lines OP and OR. [4]

Answers: **(i)** Proof; **(ii)** 76.3 cm²; **(iii)** 35.1 cm.

N12/12/Q11

27

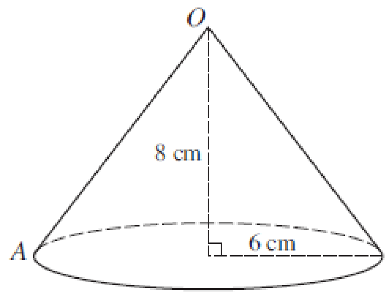


Fig. 1

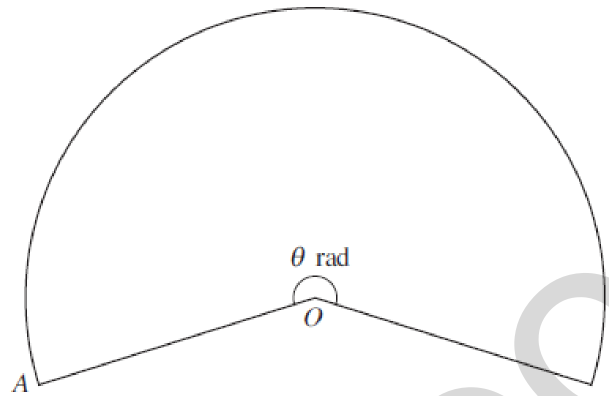


Fig. 2

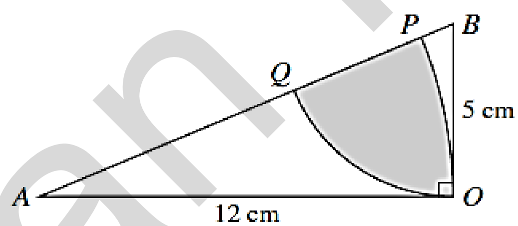
Fig. 1 shows a hollow cone with no base, made of paper. The radius of the cone is 6 cm and the height is 8 cm. The paper is cut from A to O and opened out to form the sector shown in Fig. 2. The circular bottom edge of the cone in Fig. 1 becomes the arc of the sector in Fig. 2. The angle of the sector is θ radians. Calculate

(i) the value of θ, [4]

(ii) the area of paper needed to make the cone. [2]

Answer: **(i)** 1.2π, **(ii)** 60π. (or decimal equivalents) **N13/12/Q2**

28



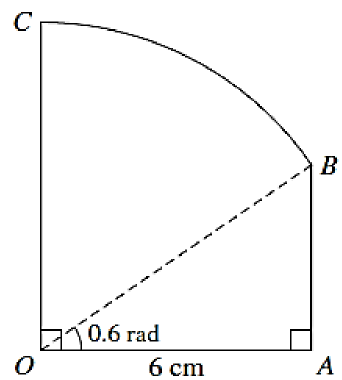
The diagram shows a triangle AOB in which OA is 12 cm, OB is 5 cm and angle AOB is a right angle. Point P lies on AB and OP is an arc of a circle with centre A. Point Q lies on AB and OQ is an arc of a circle with centre B.

(i) Show that angle BAO is 0.3948 radians, correct to 4 decimal places. [1]

(ii) Calculate the area of the shaded region. [5]

Answer: 13.1 **N14/12/Q2**

29



The diagram shows a metal plate $OABC$, consisting of a right-angled triangle OAB and a sector OBC of a circle with centre O. Angle $AOB = 0.6$ radians, $OA = 6$ cm and OA is perpendicular to OC.

(i) Show that the length of OB is 7.270 cm, correct to 3 decimal places. [1]

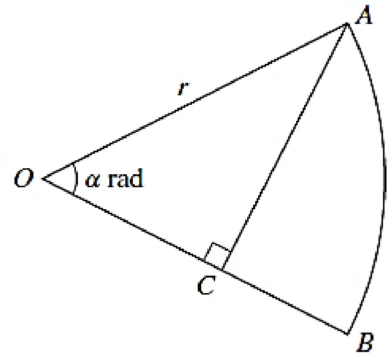
(ii) Find the perimeter of the metal plate. [3]

(iii) Find the area of the metal plate. [3]

Answers: **(ii)** 24.4, **(iii)** 38.0 **N15/12/Q5**

Homework: Circular Measure – Variants 11 & 13

1



In the diagram, OAB is a sector of a circle with centre O and radius r. The point C on OB is such that angle ACO is a right angle. Angle AOB is α radians and is such that AC divides the sector into two regions of equal area.

(i) Show that $\sin\alpha\cos\alpha = \frac{1}{2}\alpha$. [4]

It is given that the solution of the equation in part **(i)** is $\alpha = 0.9477$, correct to 4 decimal places.

(ii) Find the ratio

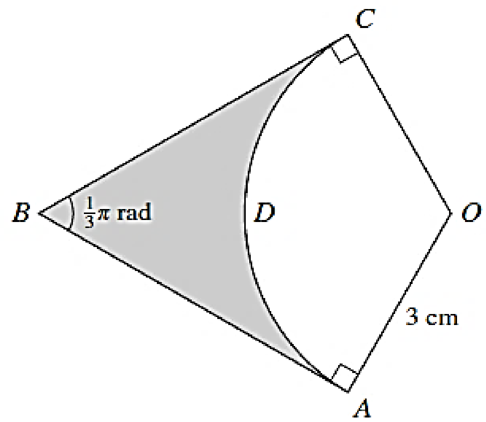
perimeter of region OAC : perimeter of region ACB,

giving your answer in the form $k : 1$, where k is given correct to 1 decimal place. [5]

(iii) Find angle AOB in degrees. [1]

- *Answers*: **(ii)** 1.1:1; **(iii)** 54.3°. **13/J15/11**

2



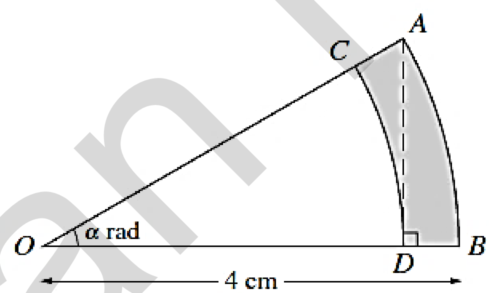
In the diagram, $OADC$ is a sector of a circle with centre O and radius 3 cm. AB and CB are tangents to the circle and angle $ABC = \frac{1}{3}\pi$ radians. Find, giving your answer in terms of $\sqrt{3}$ and π,

(i) the perimeter of the shaded region, [3]

(ii) the area of the shaded region. [3]

Answers: (i) $6\sqrt{3} + 2\pi$; (ii) $9\sqrt{3} - 3\pi$ 13/N14/2

3



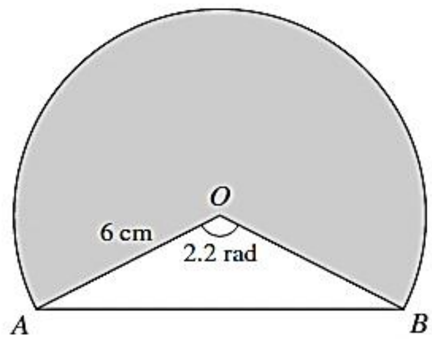
In the diagram, AB is an arc of a circle with centre O and radius 4 cm. Angle AOB is α radians. The point D on OB is such that AD is perpendicular to OB. The arc DC, with centre O, meets OA at C.

(i) Find an expression in terms of α for the perimeter of the shaded region $ABDC$. [4]

(ii) For the case where $\alpha = \frac{1}{6}\pi$, find the area of the shaded region $ABDC$, giving your answer in the form $k\pi$, where k is a constant to be determined. [4]

Answers: (i) $8 + 4\alpha\cos\alpha + 4\alpha - 8\cos\alpha$ (ii) $\frac{\pi}{3}$ or $k = \frac{1}{3}$ 11/N14/8

4



The diagram shows part of a circle with centre O and radius 6 cm. The chord AB is such that angle $AOB = 2.2$ radians. Calculate

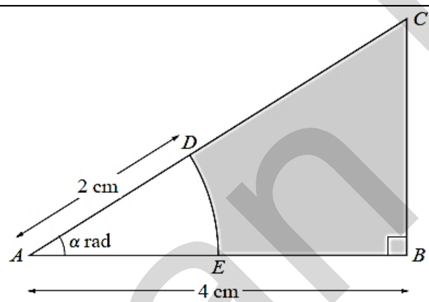
(i) the perimeter of the shaded region, [3]

(ii) the ratio of the area of the shaded region to the area of the triangle AOB, giving your answer in the form $k : 1$. [3]

Answers: **(i)** 36.5; **(ii)** 5.05.

13/J14/3

5



The diagram shows triangle ABC in which AB is perpendicular to BC. The length of AB is 4 cm and angle CAB is α radians. The arc DE with centre A and radius 2 cm meets AC at D and AB at E. Find, in terms of α,

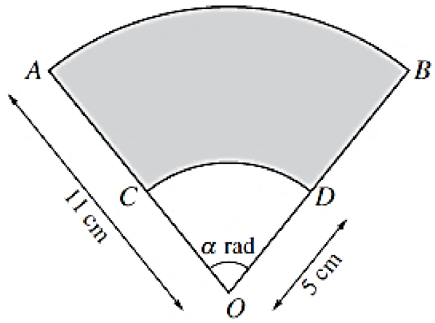
(i) the area of the shaded region, [3]

(ii) the perimeter of the shaded region. [3]

Answers: **(i)** $8\tan\alpha - 2\alpha$; **(ii)** $4\tan\alpha + 2\alpha + \dfrac{4}{\cos\alpha}$.

11/J14/6

6



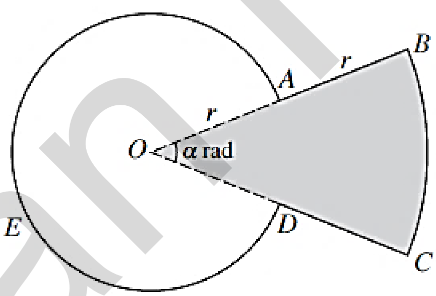
The diagram shows sector OAB with centre O and radius 11 cm. Angle $AOB = \alpha$ radians. Points C and D lie on OA and OB respectively. Arc CD has centre O and radius 5 cm.

(i) The area of the shaded region $ABDC$ is equal to k times the area of the unshaded region OCD. Find k. [3]

(ii) The perimeter of the shaded region $ABDC$ is equal to twice the perimeter of the unshaded region OCD. Find the exact value of α. [4]

Answer: 3.84, $\frac{4}{3}$ **13/N13/6**

7



The diagram shows a metal plate made by fixing together two pieces, $OABCD$ (shaded) and $OAED$ (unshaded). The piece $OABCD$ is a minor sector of a circle with centre O and radius $2r$. The piece $OAED$ is a major sector of a circle with centre O and radius r. Angle AOD is α radians. Simplifying your answers where possible, find, in terms of α, π and r,

(i) the perimeter of the metal plate, [3]

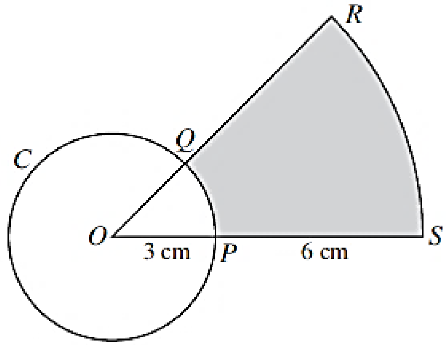
(ii) the area of the metal plate. [3]

It is now given that the shaded and unshaded pieces are equal in area.

(iii) Find α in terms of π. [2]

Answers: **(i)** $2\pi r + r\alpha + 2r$; **(ii)** $(3r^2\alpha)/2 + \pi r^2$; **(iii)** $\alpha = (2\pi)/5$. **11/N13/6**

8



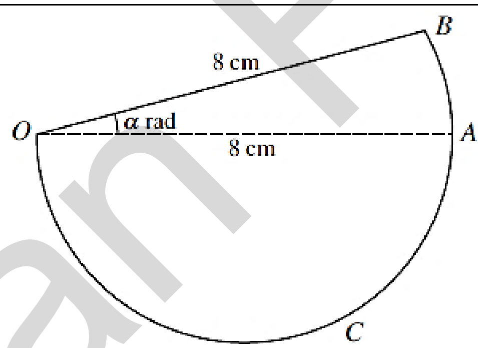
The diagram shows a circle C with centre O and radius 3 cm. The radii OP and OQ are extended to S and R respectively so that ORS is a sector of a circle with centre O. Given that $PS = 6$ cm and that the area of the shaded region is equal to the area of circle C,

(i) show that angle $POQ = \frac{1}{4}\pi$ radians, [3]

(ii) find the perimeter of the shaded region. [2]

Answers: **(ii)** 21.4 cm. **13/J13/2**

9



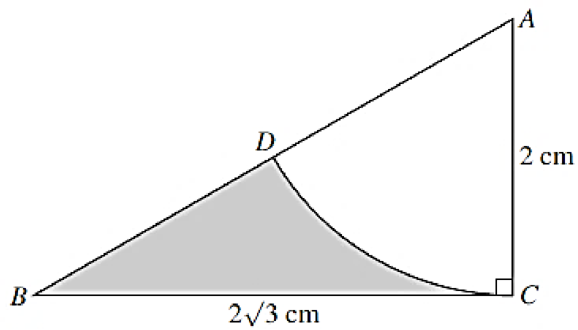
In the diagram, OAB is a sector of a circle with centre O and radius 8 cm. Angle BOA is α radians. OAC is a semicircle with diameter OA. The area of the semicircle OAC is twice the area of the sector OAB.

(i) Find α in terms of π. [3]

(ii) Find the perimeter of the complete figure in terms of π. [2]

Answers: **(i)** $\pi/8$; **(ii)** $8 + 5\pi$. **11/J13/3**

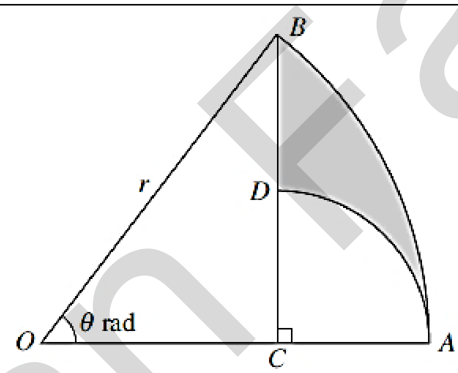
10



In the diagram, D lies on the side AB of triangle ABC and CD is an arc of a circle with centre A and radius 2 cm. The line BC is of length $2\sqrt{3}$ cm and is perpendicular to AC. Find the area of the shaded region BDC, giving your answer in terms of π and $\sqrt{3}$. [4]

Answer: $2\sqrt{3} - \frac{2\pi}{3}$. **13/N12/4**

11



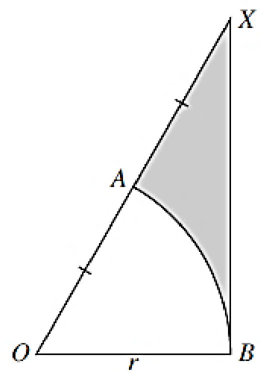
The diagram shows a sector OAB of a circle with centre O and radius r. Angle AOB is θ radians. The point C on OA is such that BC is perpendicular to OA. The point D is on BC and the circular arc AD has centre C.

(i) Find AC in terms of r and θ. [1]

(ii) Find the perimeter of the shaded region ABD when $\theta = \frac{1}{3}\pi$ and $r = 4$, giving your answer as an exact value. [6]

Answers: **(i)** $r - r\cos\theta$; **(ii)** $7\pi/3 + 2\sqrt{3} - 2$. **11/N12/6**

12



In the diagram, AB is an arc of a circle with centre O and radius r. The line XB is a tangent to the circle at B and A is the mid-point of OX.

(i) Show that angle $AOB = \frac{1}{3}\pi$ radians. [2]

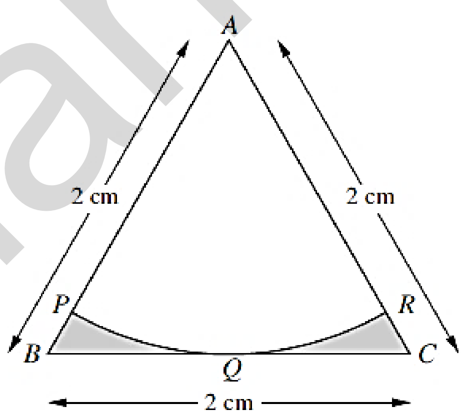
Express each of the following in terms of r, π and $\sqrt{3}$:

(ii) the perimeter of the shaded region, [3]

(iii) the area of the shaded region. [2]

Answers: **(i)** Proof **(ii)** $r + \frac{1}{3}\pi r + r\sqrt{3}$ **(iii)** $\frac{1}{2}r^2\sqrt{3} - \frac{1}{6}r^2\pi$. 13/J12/8

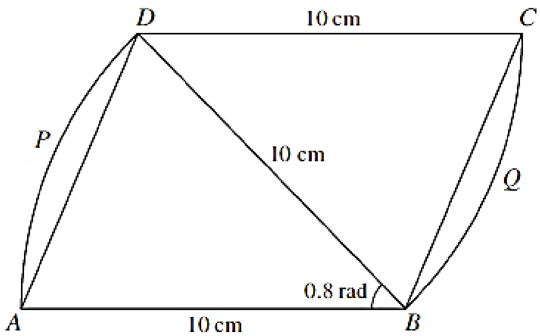
13



In the diagram, ABC is an equilateral triangle of side 2 cm. The mid-point of BC is Q. An arc of a circle with centre A touches BC at Q, and meets AB at P and AC at R. Find the total area of the shaded regions, giving your answer in terms of π and $\sqrt{3}$. [5]

Answer: $\sqrt{3} - \frac{1}{2}\pi$. 11/J12/3

14



In the diagram, $ABCD$ is a parallelogram with $AB = BD = DC = 10$ cm and angle $ABD = 0.8$ radians. APD and BQC are arcs of circles with centres B and D respectively.

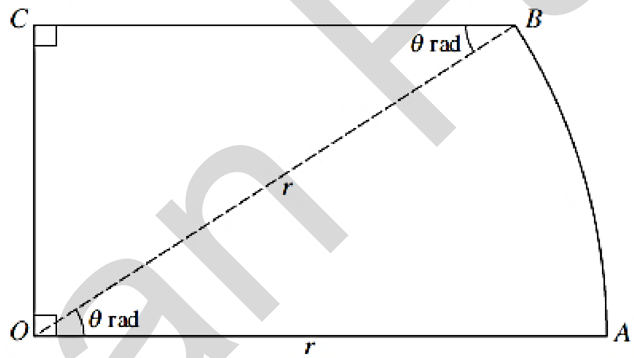
(i) Find the area of the parallelogram $ABCD$. [2]

(ii) Find the area of the complete figure $ABQCDP$. [2]

(iii) Find the perimeter of the complete figure $ABQCDP$. [2]

Answers: (i) 71.7, (ii) 80, (iii) 36 — 13/N11/4

15



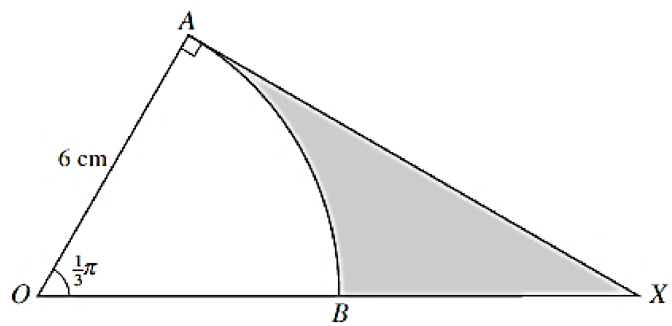
The diagram represents a metal plate $OABC$, consisting of a sector OAB of a circle with centre O and radius r, together with a triangle OCB which is right-angled at C. Angle $AOB = \theta$ radians and OC is perpendicular to OA.

(i) Find an expression in terms of r and θ for the perimeter of the plate. [3]

(ii) For the case where $r = 10$ and $\theta = \frac{1}{5}\pi$, find the area of the plate. [3]

Answers: (i) $r + r\theta + r\cos\theta + r\sin\theta$; (ii) 55.2. — 11/N11/5

16



In the diagram, AB is an arc of a circle, centre O and radius 6 cm, and angle $AOB = \frac{1}{3}\pi$ radians. The line AX is a tangent to the circle at A, and OBX is a straight line.

(i) Show that the exact length of AX is $6\sqrt{3}$ cm. [1]

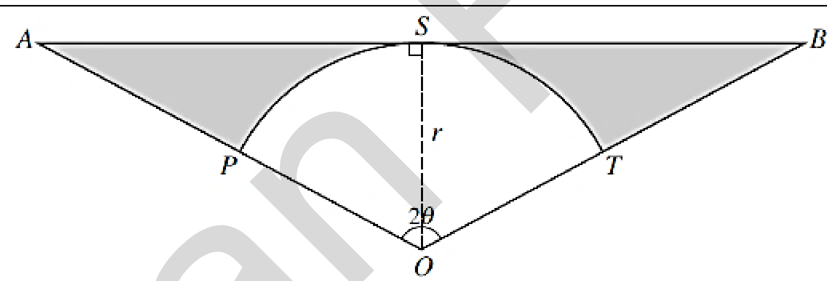
Find, in terms of π and $\sqrt{3}$,

(ii) the area of the shaded region, [3]

(iii) the perimeter of the shaded region. [4]

Answers: **(ii)** $18\sqrt{3} - 6\pi$ cm^2; **(iii)** $6\sqrt{3} + 2\pi + 6$ cm. 13/J11/7

17



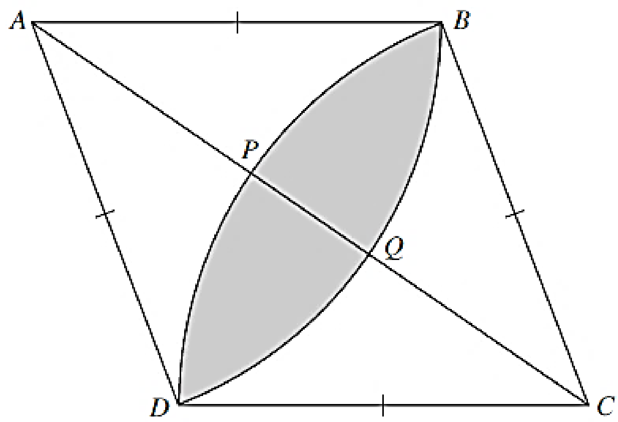
In the diagram, OAB is an isosceles triangle with $OA = OB$ and angle $AOB = 2\theta$ radians. Arc PST has centre O and radius r, and the line ASB is a tangent to the arc PST at S.

(i) Find the total area of the shaded regions in terms of r and θ. [4]

(ii) In the case where $\theta = \frac{1}{3}\pi$ and $r = 6$, find the total perimeter of the shaded regions, leaving your answer in terms of $\sqrt{3}$ and π. [5]

Answers: **(i)** $r^2(\tan\theta - \theta)$; **(ii)** $12 + 12\sqrt{3} + 4\pi$. 11/J11/9

18



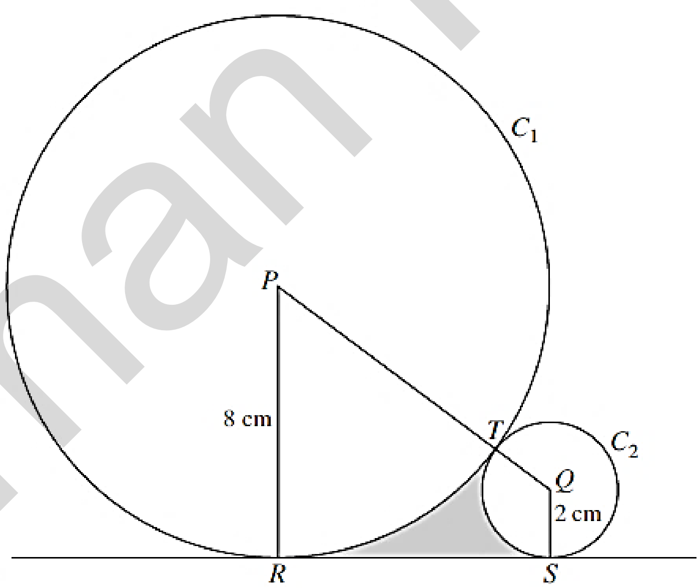
The diagram shows a rhombus $ABCD$. Points P and Q lie on the diagonal AC such that BPD is an arc of a circle with centre C and BQD is an arc of a circle with centre A. Each side of the rhombus has length 5 cm and angle $BAD = 1.2$ radians.

(i) Find the area of the shaded region $BPDQ$. [4]

(ii) Find the length of PQ. [4]

Answers: **(i)** 6.70 cm^2; **(ii)** 1.75 cm. **13/N10/8**

19



The diagram shows two circles, C_1 and C_2, touching at the point T. Circle C_1 has centre P and radius 8 cm; circle C_2 has centre Q and radius 2 cm. Points R and S lie on C_1 and C_2 respectively, and RS is a tangent to both circles.

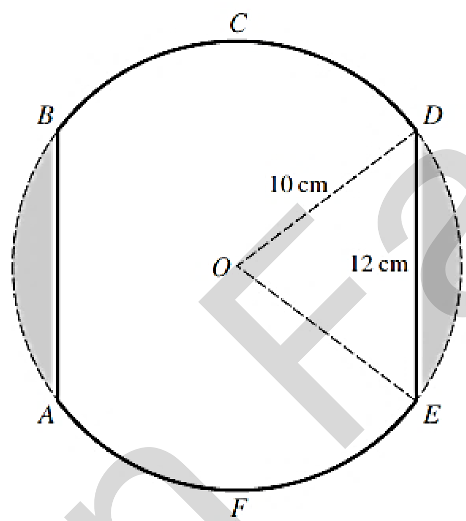
(i) Show that $RS = 8$ cm. [2]

(ii) Find angle RPQ in radians correct to 4 significant figures. [2]

(iii) Find the area of the shaded region. [4]

Answers: (ii) 0.9273; (iii) 5.90 cm^2. **11/N10/9**

20



The diagram shows a metal plate $ABCDEF$ which has been made by removing the two shaded regions from a circle of radius 10 cm and centre O. The parallel edges AB and ED are both of length 12 cm.

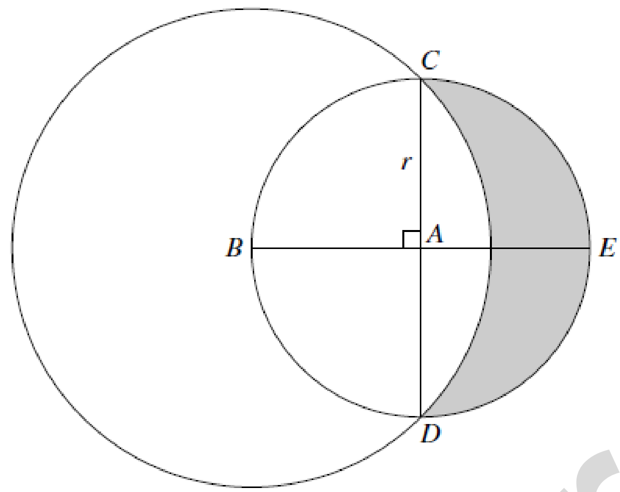
(i) Show that angle DOE is 1.287 radians, correct to 4 significant figures. [2]

(ii) Find the perimeter of the metal plate. [3]

(iii) Find the area of the metal plate. [3]

Answers: (ii) 61.1 cm; (iii) 281 or 282 cm^2. **13/J10/7**

21



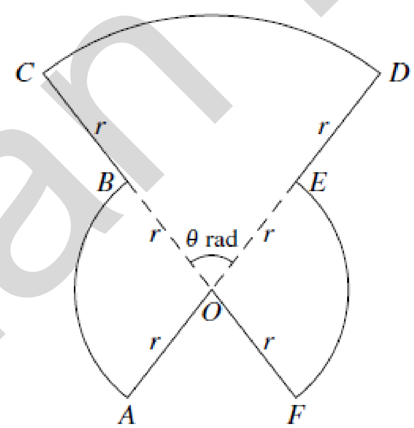
The diagram shows a circle with centre A and radius r. Diameters CAD and BAE are perpendicular to each other. A larger circle has centre B and passes through C and D.

(i) Show that the radius of the larger circle is $r\sqrt{2}$. [1]

(ii) Find the area of the shaded region in terms of r. [6]

N15/11/Q7

22



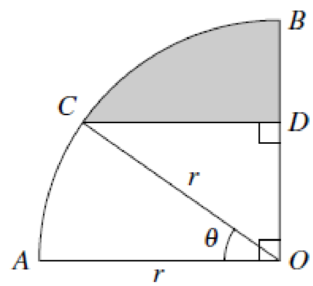
The diagram shows a metal plate $OABCDEF$ consisting of 3 sectors, each with centre O. The radius of sector COD is $2r$ and angle COD is θ radians. The radius of each of the sectors BOA and FOE is r, and $AOED$ and $CBOF$ are straight lines.

(i) Show that the area of the metal plate is $r^2(\pi + \theta)$. [3]

(ii) Show that the perimeter of the metal plate is independent of θ. [4]

N15/13/Q4

23



In the diagram, AOB is a quarter circle with centre O and radius r. The point C lies on the arc AB and the point D lies on OB. The line CD is parallel to AO and angle $AOC = \theta$ radians.

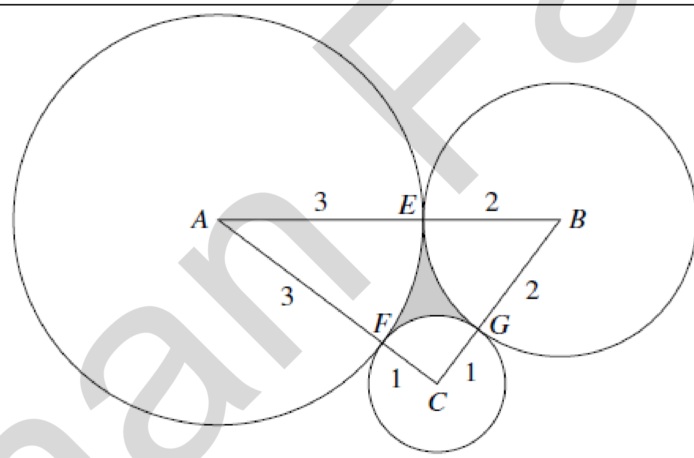
(i) Express the perimeter of the shaded region in terms of r, θ and π. [4]

(ii) For the case where $r = 5$ cm and $\theta = 0.6$, find the area of the shaded region. [3]

Answers: (i) $r\cos\theta + r - r\sin\theta + r\left(\frac{\pi}{2} - \theta\right)$ (or equivalent) (ii) 6.31

J16/11/Q7

24

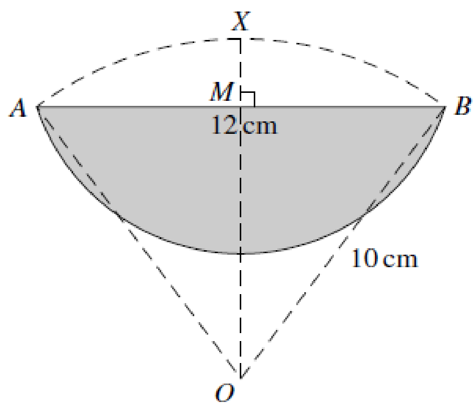


The diagram shows triangle ABC where $AB = 5$ cm, $AC = 4$ cm and $BC = 3$ cm. Three circles with centres at A, B and C have radii 3 cm, 2 cm and 1 cm respectively. The circles touch each other at points E, F and G, lying on AB, AC and BC respectively. Find the area of the shaded region EFG. [7]

Answer: 0.464.

J16/13/Q6

25



In the diagram, $OAXB$ is a sector of a circle with centre O and radius 10 cm. The length of the chord AB is 12 cm. The line OX passes through M, the mid-point of AB, and OX is perpendicular to AB. The shaded region is bounded by the chord AB and by the arc of a circle with centre X and radius XA.

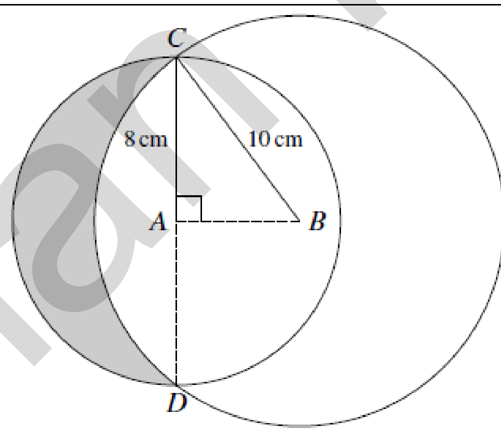
(i) Show that angle AXB is 2.498 radians, correct to 3 decimal places. [3]

(ii) Find the perimeter of the shaded region. [3]

(iii) Find the area of the shaded region. [3]

Answers: **(ii)** 27.8 cm **(iii)** 38.0 cm^2 **J17/11/Q8**

26



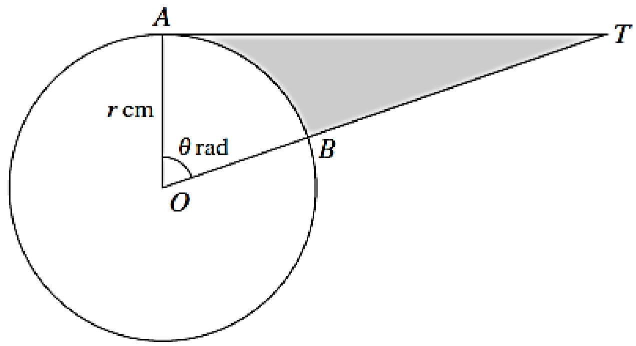
The diagram shows two circles with centres A and B having radii 8 cm and 10 cm respectively. The two circles intersect at C and D where CAD is a straight line and AB is perpendicular to CD.

(i) Find angle ABC in radians. [1]

(ii) Find the area of the shaded region. [6]

Answers : **(i)** 0.927; **(ii)** 55.8 **J17/13/Q7**

27



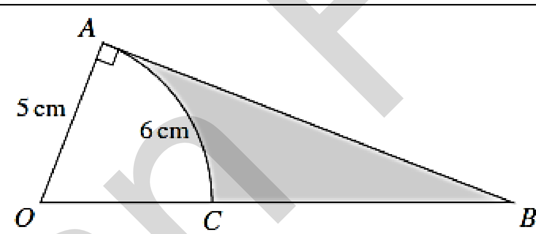
The diagram shows a circle with centre O and radius r cm. The points A and B lie on the circle and AT is a tangent to the circle. Angle $AOB = \theta$ radians and OBT is a straight line.

(i) Express the area of the shaded region in terms of r and θ. [3]

(ii) In the case where $r = 3$ and $\theta = 1.2$, find the perimeter of the shaded region. [4]

Answers: **(i)** $\frac{1}{2}r^2(\tan\theta - \theta)$ or equivalent **(ii)** 16.6 J18/11/Q6

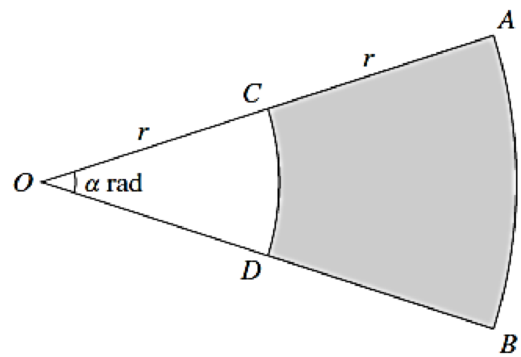
28



The diagram shows a triangle OAB in which angle $OAB = 90°$ and $OA = 5$ cm. The arc AC is part of a circle with centre O. The arc has length 6 cm and it meets OB at C. Find the area of the shaded region. [5]

Answer: 17.2. J18/13/Q5

29



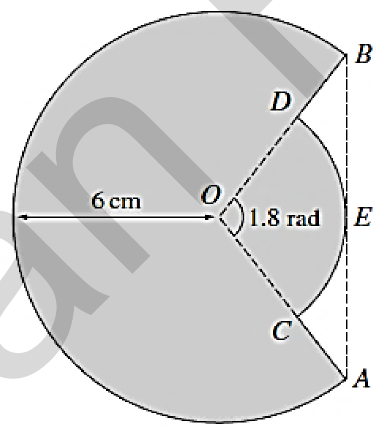
In the diagram OCA and ODB are radii of a circle with centre O and radius $2r$ cm. Angle $AOB = \alpha$ radians. CD and AB are arcs of circles with centre O and radii r cm and $2r$ cm respectively. The perimeter of the shaded region $ABDC$ is $4.4r$ cm.

(i) Find the value of α. [2]

(ii) It is given that the area of the shaded region is 30 cm^2. Find the value of r. [3]

Answers: **(i)** 0.8 **(ii)** 5 **N16/11/Q3**

30



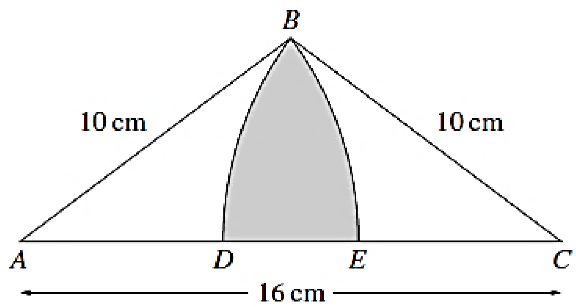
The diagram shows a major arc AB of a circle with centre O and radius 6 cm. Points C and D on OA and OB respectively are such that the line AB is a tangent at E to the arc CED of a smaller circle also with centre O. Angle $COD = 1.8$ radians.

(i) Show that the radius of the arc CED is 3.73 cm, correct to 3 significant figures. [2]

(ii) Find the area of the shaded region. [4]

Answers **(i)** 3.73; **(ii)** 93.2. **N16/13/Q5**

31



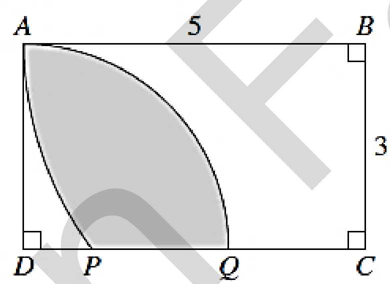
The diagram shows an isosceles triangle ABC in which $AC = 16$ cm and $AB = BC = 10$ cm. The circular arcs BE and BD have centres at A and C respectively, where D and E lie on AC.

(i) Show that angle $BAC = 0.6435$ radians, correct to 4 decimal places. [1]

(ii) Find the area of the shaded region. [5]

Answer: **(ii)** 16.4 **N17/11/Q5**

32



The diagram shows a rectangle $ABCD$ in which $AB = 5$ units and $BC = 3$ units. Point P lies on DC and AP is an arc of a circle with centre B. Point Q lies on DC and AQ is an arc of a circle with centre D.

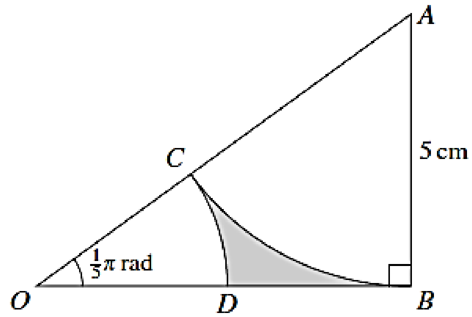
(i) Show that angle $ABP = 0.6435$ radians, correct to 4 decimal places. [1]

(ii) Calculate the areas of the sectors BAP and DAQ. [3]

(iii) Calculate the area of the shaded region. [3]

Answers: **(ii)** Sector BAP = 8.04, sector DAQ = 7.07 **(iii)** 6.11 **N17/13/Q7**

33

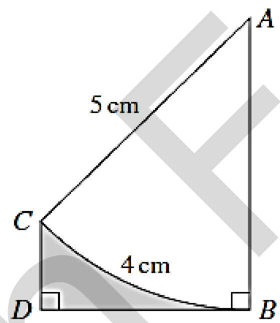


The diagram shows a triangle OAB in which angle ABO is a right angle, angle $AOB = \frac{1}{5}\pi$ radians and $AB = 5$ cm. The arc BC is part of a circle with centre A and meets OA at C. The arc CD is part of a circle with centre O and meets OB at D. Find the area of the shaded region. [8]

Answer: 1.56 – 1.57

N18/11/Q9

34



The diagram shows an arc BC of a circle with centre A and radius 5 cm. The length of the arc BC is 4 cm. The point D is such that the line BD is perpendicular to BA and DC is parallel to BA.

(i) Find angle BAC in radians. [1]

(ii) Find the area of the shaded region BDC. [5]

N18/11/Q3

Binomial Expansion

Example 1

Expand $(3 + 2x)^5$.

Example 2

Expand $(5x - 2)^4$.

Example 3

Use the binomial expansion to write down the first four terms, in ascending powers of x, of $(1 - 3x)^7$. Simplify the terms.

Example 4

Find the first three terms, in descending powers of x, in the expansion of $(4x - 3)^5$.

Example 5

Find the first five terms, in descending powers of x, in the expansion of $(x - 2)\left(x + \frac{3}{x}\right)^9$.

Example 6

The first three terms in the expansion of $\left(ax + \frac{b}{x}\right)^6$ where $a > 0$, in descending powers of x, are $64x^6 - 576x^4 + cx^2$. Find the values of a, b and c.

Example 7

Find the coefficient of x^2 in the expansion of $(2 + 3x)^7$.

Example 8

Find the coefficient of x in the expansion of $\left(2x - \frac{3}{x}\right)^5$.

Example 9

Example 7.4 *The independent term in the expansion of* $\left(px + \frac{3}{x^2}\right)^9$ *is* $\frac{28}{9}$. *Calculate the value of* p.

Example 10

November 2016/11 Question 2

Find the term independent of x in the expansion of $\left(2x + \frac{1}{2x^3}\right)^8$. [4]

Example 11

June 2016/11 Question 1

Find the term independent of x in the expansion of $\left(x - \frac{3}{2x}\right)^6$. [3]

Example 12

June 2016/13 Question 1

Find the coefficient of x in the expansion of $\left(\frac{1}{x} + 3x^2\right)^5$. [3]

Example 13

November 2015/12 Question 2

In the expansion of $(x + 2k)^7$, where k is a non-zero constant, the coefficients of x^4 and x^5 are equal. Find the value of k. [4]

Example 14

November 2016/13 Question 2

The coefficient of x^3 in the expansion of $(1 - 3x)^6 + (1 + ax)^5$ is 100. Find the value of the constant a. [4]

Example 15

a) Find the first three terms in the expansion in descending powers of x of $\left(2x - \frac{4}{x}\right)^6$.

b) Hence determine the coefficient of x^3 in the expansion of $(1 + 2x)\left(2x - \frac{4}{x}\right)^6$.

Example 16

(i) Find the first three terms in the expansion of $(2 - x)^6$ in ascending powers of x.

(ii) Find the value of k for which there is no term in x^2 in the expansion of $(1 + kx)(2 - x)^6$.

[Cambridge AS & A Level Mathematics 9709, Paper 1 Q4 June 2005]

Example 18

2020 Specimen Paper 1 Question 6

(a) Find the coefficients of x^2 and x^3 in the expansion of $(2-x)^6$. [3]

(b) Hence find the coefficient of x^3 in the expansion of $(3x+1)(2-x)^6$. [2]

Example 19

Given that the coefficient of x^3 in the expansion of $(1 + ax + 2x^2)(2 - x)^7$ is 560, determine the value of a.

Example 20

Example 7.6 *(i) Find the first 3 terms in the expansion of* $(1+ax)^6$ *in ascending powers of* x.

(ii) Given that there is no term in x *in the expansion of* $(1-2x)^2(1+ax)^6$, *find the value of* a.

(iii) For this value of a, *find the coefficient of* x^2 *in the expansion of* $(1-2x)^2(1+ax)^6$.

Example 21

June 2016/12 Question 4

Find the term that is independent of x in the expansion of

(i) $\left(x-\frac{2}{x}\right)^6$, [2]

(ii) $\left(2+\frac{3}{x^2}\right)\left(x-\frac{2}{x}\right)^6$. [4]

Example 22

(i) Find the first three terms in the expansion of $(1 + ax)^5$ in ascending powers of x.

(ii) Given that there is no term in x in the expansion of $(1 - 2x)(1 + ax)^5$, find the value of the constant a.

(iii) For this value of a, find the coefficient of x^2 in the expansion of $(1 - 2x)(1 + ax)^5$.

[Cambridge AS & A Level Mathematics 9709, Paper 12 Q6 June 2010]

Example 23

November 2016/12 Question 4

In the expansion of $(3-2x)\left(1+\frac{x}{2}\right)^n$, the coefficient of x is 7. Find the value of the constant n and hence find the coefficient of x^2. [6]

Example 24

The first three terms in the expansion of $(2+ax)^n$, in ascending powers of x, are $32-40x+bx^2$. Find the values of the constants n, a and b.

[Cambridge AS & A Level Mathematics 9709, Paper 1 Q4 June 2006]

Example 25

November 2015/11 Question 1

In the expansion of $\left(1-\frac{2x}{a}\right)(a+x)^5$, where a is a non-zero constant, show that the coefficient of x^2 is zero. [3]

Homework: Binomial Expansion Variant 12

1 Find the value of the coefficient of $\frac{1}{x}$ in the expansion of $\left(2x - \frac{1}{x}\right)^5$. [3]

Answer: -40. J03/Q1

2 Find the coefficient of x^3 in the expansion of

(i) $(1 + 2x)^6$, [3]

(ii) $(1 - 3x)(1 + 2x)^6$. [3]

Answers: (i)160; (ii) -20. J04/Q4

3 (i) Find the first 3 terms in the expansion of $(2 - x)^6$ in ascending powers of x. [3]

(ii) Find the value of k for which there is no term in x^2 in the expansion of $(1 + kx)(2 - x)^6$. [2]

Answers: (i) $64 - 192x + 240x^2$; (ii) 1.25. J05/Q4

4 The first three terms in the expansion of $(2 + ax)^n$, in ascending powers of x, are $32 - 40x + bx^2$. Find the values of the constants n, a and b. [5]

Answers: $n = 5$, $a = -\frac{1}{2}$, $b = 20$. J06/Q4

5 (i) Find the first 3 terms in the expansion, in ascending powers of x, of $(2 + x^2)^5$. [3]

(ii) Hence find the coefficient of x^4 in the expansion of $(1 + x^2)^2(2 + x^2)^5$. [3]

Answers: (i) $32 + 80x^2 + 80x^4$; (ii) 272. J08/Q3

6 (i) Find the first 3 terms in the expansion of $(2 + 3x)^5$ in ascending powers of x. [3]

(ii) Hence find the value of the constant a for which there is no term in x^2 in the expansion of $(1 + ax)(2 + 3x)^5$. [2]

Answers: (i) $32 + 240x + 720x^2$; (ii) -3 J09/Q3

7 (i) Find the first 3 terms in the expansion of $(1 + ax)^5$ in ascending powers of x. [2]

(ii) Given that there is no term in x in the expansion of $(1 - 2x)(1 + ax)^5$, find the value of the constant a. [2]

(iii) For this value of a, find the coefficient of x^2 in the expansion of $(1 - 2x)(1 + ax)^5$. [3]

Answers: (i) $1+5ax+10a^2x^2$; (ii) 0.4; (iii) -2.4. J10/12/Q6

8 Find the value of the term which is independent of x in the expansion of $\left(x+\frac{3}{x}\right)^4$. [3]

Answer: 54. N02/Q1

9 Find the coefficient of x in the expansion of $\left(3x-\frac{2}{x}\right)^5$. [4]

Answer: 1080. N04/Q1

10 Find the coefficient of x^2 in the expansion of $\left(x+\frac{2}{x}\right)^6$. [3]

Answer: 60. N06/Q1

11 **(i)** Find the first three terms in the expansion of $(2+u)^5$ in ascending powers of u. [3]

(ii) Use the substitution $u=x+x^2$ in your answer to part **(i)** to find the coefficient of x^2 in the expansion of $(2+x+x^2)^5$. [2]

Answers: (i) $32+80u+80u^2$; (ii) 160. N07/Q3

12 Find the value of the coefficient of x^2 in the expansion of $\left(\frac{x}{2}+\frac{2}{x}\right)^6$. [3]

Answer: $3\frac{3}{4}$. N08/Q1

13 **(i)** Find, in terms of the non-zero constant k, the first 4 terms in the expansion of $(k+x)^8$ in ascending powers of x. [3]

(ii) Given that the coefficients of x^2 and x^3 in this expansion are equal, find the value of k. [2]

Answers: (i) $k^8+8k^7x+28k^6x^2+56k^5x^3$; (ii) 2. N09/12/Q2

14 **(i)** Find the first 3 terms in the expansion, in ascending powers of x, of $(1-2x^2)^8$. [2]

(ii) Find the coefficient of x^4 in the expansion of $(2-x^2)(1-2x^2)^8$. [2]

Answers: (i) $1-16x^2+112x^4$; (ii) 240. N10/12/Q1

15 (i) Find the terms in x^2 and x^3 in the expansion of $\left(1-\frac{3}{2}x\right)^6$. [3]

(ii) Given that there is no term in x^3 in the expansion of $(k+2x)\left(1-\frac{3}{2}x\right)^6$, find the value of the constant k. [2]

Answers: **(i)** $\frac{135x^2}{4}$, $-\frac{540x^3}{8}$; **(ii)** 1. J11/12/Q2

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16 (i) Find the first 3 terms in the expansion of $(2-y)^5$ in ascending powers of y. [2]

(ii) Use the result in part (i) to find the coefficient of x^2 in the expansion of $\left(2-(2x-x^2)\right)^5$. [3]

Answers: **(i)** $32 - 80y + 80y^2$; **(ii)** 400 N11/12/Q1

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17 The coefficient of x^3 in the expansion of $(a+x)^5 + (2-x)^6$ is 90. Find the value of the positive constant a. [5]

Answers: **(i)** −8½; **(ii)** −12**i** + 24**j** + 8**k** N11/12/Q3

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18 Find the coefficient of x^2 in the expansion of

(i) $\left(2x-\frac{1}{2x}\right)^6$, [2]

(ii) $(1+x^2)\left(2x-\frac{1}{2x}\right)^6$. [3]

Answers: **(i)** 60. **(ii)** 40. J13/12/Q2

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19 Find the coefficient of x^2 in the expansion of $(1+x^2)\left(\frac{x}{2}-\frac{4}{x}\right)^6$. [5]

Answer: −145 J14/12/Q2

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20 (i) Find the coefficients of x^2 and x^3 in the expansion of $(2-x)^6$. [3]

(ii) Find the coefficient of x^3 in the expansion of $(3x+1)(2-x)^6$. [2]

Answer: **(i)** 240, −160 ; **(ii)** 560. J15/12/Q3

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21 In the expansion of $\left(x^2-\frac{a}{x}\right)^7$, the coefficient of x^5 is −280. Find the value of the constant a. [3]

Answer: $a = 2$. N12/12/Q1

22 (i) Find the first 3 terms, in ascending powers of x, in the expansion of $(1+x)^5$. [2]

The coefficient of x^2 in the expansion of $\left(1+(px+x^2)\right)^5$ is 95.

(ii) Use the answer to part (i) to find the value of the positive constant p. [3]

Answer: $p = 3$ **N14/12/Q3**

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23 In the expansion of $(x+2k)^7$, where k is a non-zero constant, the coefficients of x^4 and x^5 are equal. Find the value of k. [4]

Answer: $k = 0.3$ **N15/12/Q2**

Homework: Binomial Expansion – Variants 11 & 13

1 (i) Write down the first 4 terms, in ascending powers of x, of the expansion of $(a-x)^5$. [2]

(ii) The coefficient of x^3 in the expansion of $(1-ax)(a-x)^5$ is -200. Find the possible values of the constant a. [4]

Answers: **(i)** $a^5 - 5a^4x + 10a^3x^2 - 10a^2x^3$; **(ii)** $a = \pm 2$. **13/J15/3**

2 (i) Find the first three terms, in ascending powers of x, in the expansion of

(a) $(1-x)^6$, [2]

(b) $(1+2x)^6$. [2]

(ii) Hence find the coefficient of x^2 in the expansion of $[(1-x)(1+2x)]^6$. [3]

Answers: **(i)(a)** $1-6x+15x^2$ **(b)** $1+12x+60x^2$ **(ii)** 3 **11/J15/3**

3 In the expansion of $(2+ax)^6$, the coefficient of x^2 is equal to the coefficient of x^3. Find the value of the non-zero constant a. [4]

Answer: $\frac{3}{2}$ **13/N14/1**

4 In the expansion of $(2+ax)^7$, the coefficient of x is equal to the coefficient of x^2. Find the value of the non-zero constant a. [3]

Answer: $a = \frac{2}{3}$ **11/N14/1**

5 Find the coefficient of x in the expansion of $\left(x^2 - \frac{2}{x}\right)^5$. [3]

Answer: –80. **13/J14/1**

6 Find the term independent of x in the expansion of $\left(4x^3 + \frac{1}{2x}\right)^8$. [4]

Answer: 7. **11/J14/3**

7 (i) Find the coefficient of x^8 in the expansion of $(x+3x^2)^4$. [1]

(ii) Find the coefficient of x^8 in the expansion of $(x+3x^2)^5$. [3]

(iii) Hence find the coefficient of x^8 in the expansion of $[1+(x+3x^2)]^5$. [4]

Answer: 81, 270, 675 13/N13/8

8 (i) Find the first three terms when $(2+3x)^6$ is expanded in ascending powers of x. [3]

(ii) In the expansion of $(1+ax)(2+3x)^6$, the coefficient of x^2 is zero. Find the value of a. [2]

Answers: (i) $64+576x+2160x^2$; (ii) -3.75. 11/N13/1

9 (i) Find the first three terms in the expansion of $(2+ax)^5$ in ascending powers of x. [3]

(ii) Given that the coefficient of x^2 in the expansion of $(1+2x)(2+ax)^5$ is 240, find the possible values of a. [3]

Answers: (i) $32+80ax+80a^2x^2$; (ii) $a=-3$, $a=1$. 13/J13/4

10 (i) In the expression $(1-px)^6$, p is a non-zero constant. Find the first three terms when $(1-px)^6$ is expanded in ascending powers of x. [2]

(ii) It is given that the coefficient of x^2 in the expansion of $(1-x)(1-px)^6$ is zero. Find the value of p. [3]

Answers: (i) $1-6px+15p^2x^2$; (ii) $-2/5$. 11/J13/2

11 Find the coefficient of x^3 in the expansion of $(2-\frac{1}{2}x)^7$. [3]

Answer: -70. 13/N12/1

12 (i) Find the first 3 terms in the expansion of $(2x-x^2)^6$ in ascending powers of x. [3]

(ii) Hence find the coefficient of x^8 in the expansion of $(2+x)(2x-x^2)^6$. [2]

Answers: (i) $64x^6-192x^7+240x^8$; (ii) 288. 11/N12/4

13 The first three terms in the expansion of $(1-2x)^2(1+ax)^6$, in ascending powers of x, are $1-x+bx^2$. Find the values of the constants a and b. [6]

Answers: $a=½$, $b=-4¼$ 13/J12/3

14 Find the coefficient of x^6 in the expansion of $\left(2x^3-\frac{1}{x^2}\right)^7$. [4]

Answer: -560 11/J12/2

15 The coefficient of x^2 in the expansion of $(k + \frac{1}{3}x)^5$ is 30. Find the value of the constant k. [3]

Answer: 3 — 13/N11/1

16 Find the term independent of x in the expansion of $\left(2x + \frac{1}{x^2}\right)^6$. [3]

Answer: 240. — 11/N11/1

17 The coefficient of x^3 in the expansion of $(a+x)^5 + (1-2x)^6$, where a is positive, is 90. Find the value of a. [5]

Answer: 5. — 13/J11/1

18 Find the coefficient of x in the expansion of $\left(x + \frac{2}{x^2}\right)^7$. [3]

Answer: 84. — 11/J11/1

19 Find the term independent of x in the expansion of $\left(x - \frac{1}{x^2}\right)^9$. [3]

Answer: −84. — 13/N10/1

20 (i) Find the first three terms, in descending powers of x, in the expansion of $\left(x - \frac{2}{x}\right)^6$. [3]

(ii) Find the coefficient of x^4 in the expansion of $(1+x^2)\left(x - \frac{2}{x}\right)^6$. [2]

Answers: (i) $x^6 - 12x^4 + 60x^2$; (ii) 48. — 13/J10/2

21 (i) Find the first 3 terms in the expansion of $\left(2x - \frac{3}{x}\right)^5$ in descending powers of x. [3]

(ii) Hence find the coefficient of x in the expansion of $\left(1 + \frac{2}{x^2}\right)\left(2x - \frac{3}{x}\right)^5$. [2]

Answers: (i) $32x^5 - 240x^3 + 720x$; (ii) 240. — 11/J10/2

22 In the expansion of $\left(1 - \frac{2x}{a}\right)(a+x)^5$, where a is a non-zero constant, show that the coefficient of x^2 is zero. [3]

N15/11/Q1

23 Find the coefficient of x in the expansion of $\left(\frac{x}{3}+\frac{9}{x^2}\right)^7$. [4]

N15/13/Q2

24 Find the term independent of x in the expansion of $\left(x-\frac{3}{2x}\right)^6$. [3]

Answer: –67.5

J16/11/Q1

25 Find the coefficient of x in the expansion of $\left(\frac{1}{x}+3x^2\right)^5$. [3]

Answer: 90.

J16/13/Q1

26 The coefficients of x^2 and x^3 in the expansion of $(3-2x)^6$ are a and b respectively. Find the value of $\frac{a}{b}$. [4]

Answer: $\frac{-9}{8}$

J17/11/Q1

27 The coefficients of x and x^2 in the expansion of $(2+ax)^7$ are equal. Find the value of the non-zero constant a. [3]

Answer : 2/3.

J17/13/Q1

28 (i) Find the first three terms in the expansion, in ascending powers of x, of $(1-2x)^5$. [2]

(ii) Given that the coefficient of x^2 in the expansion of $(1+ax+2x^2)(1-2x)^5$ is 12, find the value of the constant a. [3]

Answers: **(i)** $1-10x+40x^2$, **(ii)** $a=3$

J18/11/Q1

29 Find the coefficient of $\frac{1}{x}$ in the expansion of $\left(x-\frac{2}{x}\right)^5$. [3]

Answer: –80.

J18/13/Q2

30 Find the term independent of x in the expansion of $\left(2x+\frac{1}{2x^3}\right)^8$. [4]

Answer: 448

N16/11/Q2

31 The coefficient of x^3 in the expansion of $(1-3x)^6 + (1+ax)^5$ is 100. Find the value of the constant a. [4]

Answer: $a = 4$. N16/13/Q2

32 (i) Find the term independent of x in the expansion of $\left(\frac{2}{x} - 3x\right)^6$. [2]

(ii) Find the value of a for which there is no term independent of x in the expansion of

$$(1 + ax^2)\left(\frac{2}{x} - 3x\right)^6.$$ [3]

Answer: 50.8°, 129.2°, 230.8°, 309.2°. N16/13/Q3

33 Find the coefficient of $\frac{1}{x^3}$ in the expansion of $\left(x - \frac{2}{x}\right)^7$. [3]

Answer: –672 N18/13/Q1

Arithmetic and Geometric Progressions

Arithmetic Progressions

An arithmetic progression with n terms can be written as

$$a, (a + d), (a + 2d), (a + 3d), \ldots, \ldots, \ldots, [a + (n - 1)d]$$

N^th term of an Arithmetic Progression

The ***n*th term** of an arithmetic progression can be written as

$$\boldsymbol{u_n = a + (n - 1)d}$$

<u>Example 1</u>

The 1st term of an arithmetic progression is -6 and the common difference is 5.
Find the 4th term and the nth term.

<u>Example 2</u>

The nth term of an arithmetic progression is $7n - 2$. Find the 1st term and the common difference, d.

<u>Example 3</u>

The 9th term of an arithmetic progression is 8 and the 4th term is 18.
Find the 1st term and the common difference, d.

<u>Example 4</u>

The 3rd term of an arithmetic progression is 1 and the 6th term is 10.
Find the 4th term.

Sum of an Arithmetic Progression

The **sum of an arithmetic progression** can be written as

$$S_n = \frac{n}{2}(a + l) \quad \text{or} \quad S_n = \frac{n}{2}[2a + (n - 1)d]$$

<u>Example 5</u>

The 5th term of an arithmetic progression is 21 and the sum of the first six terms is 90.
Find the 18th term.

Example 6

The first term of an arithmetic progression is 6 and the fifth term is 12. The progression has n terms and the sum of all the terms is 90. Find the value of n.

[Cambridge AS & A Level Mathematics 9709, Paper 1 Q3 November 2008]

Example 7

2020 Specimen Paper 1 Question 3

An arithmetic progression has first term 7. The nth term is 84 and the $(3n)$th term is 245.

Find the value of n. [4]

Example 8

Mo has seven pieces of wood. Each piece of wood has a different length. The lengths are in arithmetic progression. The length of the largest piece of wood is 5 times the length of the smallest piece of wood. The total length of all seven pieces of wood is 630 cm. Work out the length of the largest piece.

Example 9

The first three terms of an arithmetic progression are $(m - 3)$, $(m + 1)$ and $(5m + 5)$.

a) Work out the value of m.

b) Work out the sum of the first 10 terms.

Example 10

The sum of the first n terms of an arithmetic progression is $n^2 + 5n$.

a) Find the nth term. **b)** Write down the first four terms.

Example 11

The sum of the first n terms of an arithmetic progression is $n^2 - 3n$.
Write down the 10th term.

Example 12

Jamila starts a part-time job on a salary of \$9000 per year, and this increases by an annual increment of \$1000. Assuming that, apart from the increment, Jamila's salary does not increase, find

(i) her salary in the 12th year

(ii) the length of time she has been working when her total earnings are \$100 000.

Example 13

Example 7.8 *Find the sum of all the multiples of* 6 *between* 200 *and* 300 *inclusive.*

Example 14

Example 7.9 *An arithmetic progression contains 20 terms. Given that the ninth term is 29 and the sum of all terms in the progression is 930. Calculate*

(i) the first term and the common difference,

(ii) the sum of all positive terms in the progression.

Example 7.10 *A circle is divided into 6 sectors in such a way that the angles of the sectors are in arithmetic progression. The angle of the largest sector is 5 times the angle of the smallest sector. Given that the radius of the circle is 9 cm, find the area of the smallest sector.*

Geometric Progressions

A geometric progression with n terms can be written as

$a, ar, ar^2, ar^3, \ldots, \ldots, \ldots, ar^{n-1}$

The ***n*th term** of a geometric progression can be written as ar^{n-1}.

Example 1

The 3rd term of a geometric progression is 9 and the common ratio is -3.
Find the 6th term and the nth term.

Example 2

The nth term of a geometric progression is $10\left(\frac{1}{2}\right)^{n-1}$. Find the 1st term and the common ratio.

The **sum of a geometric progression** can be written as $S_n = \frac{a(1-r^n)}{1-r}$, $r \neq 1$.

Example 3

The 3rd term of a geometric progression is 16 and the 6th term is -128.
Find the 1st term and the sum of the first seven terms.

Example 4

Mabintou deposits \$100 in a bank account at the start of each year.
She earns 4% compound interest.

a) Work out how much money she has in her bank account at the end of the 3rd year.

b) Work out how much money she has in her account at the start of the 30th year after she has made her annual deposit.

Example 5

An arithmetic progression has first term a and common difference d.
The 1st, 3rd and 13th terms of the arithmetic progression form the first three terms of a geometric progression with common ratio r.

a) Express d in terms of a, and find the value of r.

b) The 4th term in the geometric progression also appears in the arithmetic progression. Determine which term it is.

Example 6

$x + 5$, x and $x - 4$ are three consecutive terms of a geometric progression.
Find the value of x.

Example 7

The sum to n terms of a geometric progression is $3^n - 1$.
Find the first four terms and the common ratio, r.

Infinite Geometric Progressions

$$S_\infty = \frac{a}{1-r} \text{ provided } |r| < 1$$

Example 8

The nth term of a geometric progression is $81\left(-\frac{1}{3}\right)^n$. Find the sum to infinity.

Example 9

a) Write the recurring decimal $0.4\dot{5}$ as the sum of a geometric progression.

b) Use your answer to part **(a)** to show that $0.4\dot{5}$ can be written as $\frac{41}{90}$.

Example 10

A geometric progression has the sum to infinity equal to twice the 1st term.
Find the common ratio, r.

Example 11

Example 7.13 *A geometric progression has common ratio* $-\frac{1}{3}$ *and the sum of the first 3 terms is 105. Find*

(i) *the first term of the progression,*

(ii) *the sum to infinity.*

Example 12

Example 7.14 *The third, fourth and fifth consecutive terms of a geometric progression are* $2k+3$, $k+6$ *and* k. *Given that all terms of the geometric progression are positive, calculate*

(i) *the value of the constant* k,

(ii) *the sum to infinity.*

Example 13

November 2016/11 Question 5

The sum of the 1st and 2nd terms of a geometric progression is 50 and the sum of the 2nd and 3rd terms is 30. Find the sum to infinity. **[6]**

Example 15

The first term of a geometric progression is 81 and the fourth term is 24. Find

(i) the common ratio of the progression

(ii) the sum to infinity of the progression.

The second and third terms of this geometric progression are the first and fourth terms respectively of an arithmetic progression.

(iii) Find the sum of the first ten terms of the arithmetic progression.

[Cambridge AS & A Level Mathematics 9709, Paper 1 Q7 June 2008]

Example 16

A progression has a second term of 96 and a fourth term of 54. Find the first term of the progression in each of the following cases:

(i) the progression is arithmetic

(ii) the progression is geometric with a positive common ratio.

[Cambridge AS & A Level Mathematics 9709, Paper 12 Q3 November 2009]

Example 17

(i) Find the sum to infinity of the geometric progression with first three terms 0.5, 0.5^3 and 0.5^5.

(ii) The first two terms in an arithmetic progression are 5 and 9. The last term in the progression is the only term which is greater than 200. Find the sum of all the terms in the progression.

[Cambridge AS & A Level Mathematics 9709, Paper 1 Q7 June 2009]

Example 18

The 1st term of an arithmetic progression is a and the common difference is d, where $d \neq 0$.

(i) Write down expressions, in terms of a and d, for the 5th term and the 15th term.

The 1st term, the 5th term and the 15th term of the arithmetic progression are the first three terms of a geometric progression.

(ii) Show that $3a = 8d$.

(iii) Find the common ratio of the geometric progression.

[Cambridge AS & A Level Mathematics 9709, Paper 1 Q4 November 2007]

Example 19

November 2016/12 Question 8

(a) A cyclist completes a long-distance charity event across Africa. The total distance is 3050 km. He starts the event on May 1st and cycles 200 km on that day. On each subsequent day he reduces the distance cycled by 5 km.

(i) How far will he travel on May 15th? [2]

(ii) On what date will he finish the event? [3]

(b) A geometric progression is such that the third term is 8 times the sixth term, and the sum of the first six terms is $31\frac{1}{2}$. Find

(i) the first term of the progression, [4]

(ii) the sum to infinity of the progression. [1]

Example 20

November 2016/13 Question 9

(a) Two convergent geometric progressions, P and Q, have the same sum to infinity. The first and second terms of P are 6 and $6r$ respectively. The first and second terms of Q are 12 and $-12r$ respectively. Find the value of the common sum to infinity. [3]

(b) The first term of an arithmetic progression is $\cos\theta$ and the second term is $\cos\theta + \sin^2\theta$, where $0 \leqslant \theta \leqslant \pi$. The sum of the first 13 terms is 52. Find the possible values of θ. [5]

Example 21

June 2016/11 Question 9

(a) The first term of a geometric progression in which all the terms are positive is 50. The third term is 32. Find the sum to infinity of the progression. [3]

(b) The first three terms of an arithmetic progression are $2\sin x$, $3\cos x$ and $(\sin x + 2\cos x)$ respectively, where x is an acute angle.

(i) Show that $\tan x = \frac{4}{3}$. [3]

(ii) Find the sum of the first twenty terms of the progression. [3]

Example 22

June 2016/12 Question 9

A water tank holds 2000 litres when full. A small hole in the base is gradually getting bigger so that each day a greater amount of water is lost.

(i) On the first day after filling, 10 litres of water are lost and this increases by 2 litres each day.

(a) How many litres will be lost on the 30th day after filling? [2]

(b) The tank becomes empty during the nth day after filling. Find the value of n. [3]

(ii) Assume instead that 10 litres of water are lost on the first day and that the amount of water lost increases by 10% on each succeeding day. Find what percentage of the original 2000 litres is left in the tank at the end of the 30th day after filling. [4]

Example 23

June 2016/13 Question 4

The 1st, 3rd and 13th terms of an arithmetic progression are also the 1st, 2nd and 3rd terms respectively of a geometric progression. The first term of each progression is 3. Find the common difference of the arithmetic progression and the common ratio of the geometric progression. [5]

Example 24

November 2015/11 Question 8

The first term of a progression is $4x$ and the second term is x^2.

(i) For the case where the progression is arithmetic with a common difference of 12, find the possible values of x and the corresponding values of the third term. [4]

(ii) For the case where the progression is geometric with a sum to infinity of 8, find the third term. [4]

Example 25

November 2015/13 Question 6

A ball is such that when it is dropped from a height of 1 metre it bounces vertically from the ground to a height of 0.96 metres. It continues to bounce on the ground and each time the height the ball reaches is reduced. Two different models, A and B, describe this.

Model A : The height reached is reduced by 0.04 metres each time the ball bounces.

Model B : The height reached is reduced by 4% each time the ball bounces.

(i) Find the total distance travelled vertically (up and down) by the ball from the 1st time it hits the ground until it hits the ground for the 21st time,

(a) using model A, [3]

(b) using model B. [3]

(ii) Show that, under model B, even if there is no limit to the number of times the ball bounces, the total vertical distance travelled after the first time it hits the ground cannot exceed 48 metres. [2]

Homework: Arithmetic and Geometric Progression Variant 12

1 In an arithmetic progression, the 1st term is -10, the 15th term is 11 and the last term is 41. Find the sum of all the terms in the progression. [5]

Answer: 542.5 . J03/Q4

2 A geometric progression has first term 64 and sum to infinity 256. Find

(i) the common ratio, [2]

(ii) the sum of the first ten terms. [2]

Answers: (ii) $\frac{3}{4}$; (ii) 242. J04/Q1

3 A geometric progression has 6 terms. The first term is 192 and the common ratio is 1.5. An arithmetic progression has 21 terms and common difference 1.5. Given that the sum of all the terms in the geometric progression is equal to the sum of all the terms in the arithmetic progression, find the first term and the last term of the arithmetic progression. [6]

Answers: 175 and 205. J05/Q6

4 Each year a company gives a grant to a charity. The amount given each year increases by 5% of its value in the preceding year. The grant in 2001 was \$5000. Find

(i) the grant given in 2011, [3]

(ii) the total amount of money given to the charity during the years 2001 to 2011 inclusive. [2]

Answers: (i) \$8140; (ii) \$71 000. J06/Q3

5 The second term of a geometric progression is 3 and the sum to infinity is 12.

(i) Find the first term of the progression. [4]

An arithmetic progression has the same first and second terms as the geometric progression.

(ii) Find the sum of the first 20 terms of the arithmetic progression. [3]

Answers: (i) 6 ; (ii) -450. J07/Q7

6 The first term of a geometric progression is 81 and the fourth term is 24. Find

(i) the common ratio of the progression, [2]

(ii) the sum to infinity of the progression. [2]

The second and third terms of this geometric progression are the first and fourth terms respectively of an arithmetic progression.

(iii) Find the sum of the first ten terms of the arithmetic progression. [3]

Answers: (i) $\frac{2}{3}$; (ii) 243; (iii) 270. **J08/Q7**

7 (a) Find the sum to infinity of the geometric progression with first three terms 0.5, 0.5^3 and 0.5^5. [3]

(b) The first two terms in an arithmetic progression are 5 and 9. The last term in the progression is the only term which is greater than 200. Find the sum of all the terms in the progression. [4]

Answers: (a) ⅔ or 0.667; (b) 5150 **J09/Q7**

8 (a) Find the sum of all the multiples of 5 between 100 and 300 inclusive. [3]

(b) A geometric progression has a common ratio of $-\frac{2}{3}$ and the sum of the first 3 terms is 35. Find

(i) the first term of the progression, [3]

(ii) the sum to infinity. [2]

Answers: (a) 8200; (b)(i) 45, (ii) 27. **J10/12/Q7**

9 A geometric progression, for which the common ratio is positive, has a second term of 18 and a fourth term of 8. Find

(i) the first term and the common ratio of the progression, [3]

(ii) the sum to infinity of the progression. [2]

Answers: (i) $a = 27$, $r = \frac{2}{3}$; (ii) 81. **N02/Q2**

10 (a) A debt of $3726 is repaid by weekly payments which are in arithmetic progression. The first payment is $60 and the debt is fully repaid after 48 weeks. Find the third payment. [3]

(b) Find the sum to infinity of the geometric progression whose first term is 6 and whose second term is 4. [3]

Answers: (a) $61.50; (b) 18. **N03/Q3**

11 Find

(i) the sum of the first ten terms of the geometric progression 81, 54, 36, ... , [3]

(ii) the sum of all the terms in the arithmetic progression 180, 175, 170, ... , 25. [3]

Answers: (i) 239; (ii) 3280. **N04/Q2**

12 A small trading company made a profit of $250 000 in the year 2000. The company considered two different plans, plan A and plan B, for increasing its profits.

Under plan A, the annual profit would increase each year by 5% of its value in the preceding year. Find, for plan A,

(i) the profit for the year 2008, [3]

(ii) the total profit for the 10 years 2000 to 2009 inclusive. [2]

Under plan B, the annual profit would increase each year by a constant amount $$D$.

(iii) Find the value of D for which the total profit for the 10 years 2000 to 2009 inclusive would be the same for both plans. [3]

Answers: (i) $369 000; (ii) $3 140 000; (iii) 14 300. **N05/Q6**

13 **(a)** Find the sum of all the integers between 100 and 400 that are divisible by 7. [4]

(b) The first three terms in a geometric progression are 144, x and 64 respectively, where x is positive. Find

(i) the value of x,

(ii) the sum to infinity of the progression.

[5]

Answer: (a) 10 836; (b)(i) 96, (ii) 432. **N06/Q6**

14 The 1st term of an arithmetic progression is a and the common difference is d, where $d \neq 0$.

(i) Write down expressions, in terms of a and d, for the 5th term and the 15th term. [1]

The 1st term, the 5th term and the 15th term of the arithmetic progression are the first three terms of a geometric progression.

(ii) Show that $3a = 8d$. [3]

(iii) Find the common ratio of the geometric progression. [2]

Answers: (i) $a + 4d$, $a + 14d$; (iii) 2.5. **N07/Q4**

15 The first term of an arithmetic progression is 6 and the fifth term is 12. The progression has n terms and the sum of all the terms is 90. Find the value of n. [4]

Answer: 8. **N08/Q3**

16 A progression has a second term of 96 and a fourth term of 54. Find the first term of the progression in each of the following cases:

(i) the progression is arithmetic, [3]

(ii) the progression is geometric with a positive common ratio. [3]

Answers: (i) 117; (ii) 128. **N09/12/Q3**

17 **(a)** The first and second terms of an arithmetic progression are 161 and 154 respectively. The sum of the first m terms is zero. Find the value of m. [3]

(b) A geometric progression, in which all the terms are positive, has common ratio r. The sum of the first n terms is less than 90% of the sum to infinity. Show that $r^n > 0.1$. [3]

Answer: (i) 47. **N10/12/Q5**

18 **(a)** A circle is divided into 6 sectors in such a way that the angles of the sectors are in arithmetic progression. The angle of the largest sector is 4 times the angle of the smallest sector. Given that the radius of the circle is 5 cm, find the perimeter of the smallest sector. [6]

(b) The first, second and third terms of a geometric progression are $2k + 3$, $k + 6$ and k, respectively. Given that all the terms of the geometric progression are positive, calculate

(i) the value of the constant k, [3]

(ii) the sum to infinity of the progression. [2]

Answers: **(a)** 12.1 cm; **(b)(i)** 12, **(ii)** 81. **J11/12/Q10**

19 **(a)** An arithmetic progression contains 25 terms and the first term is −15. The sum of all the terms in the progression is 525. Calculate

(i) the common difference of the progression, [2]

(ii) the last term in the progression, [2]

(iii) the sum of all the positive terms in the progression. [2]

(b) A college agrees a sponsorship deal in which grants will be received each year for sports equipment. This grant will be \$4000 in 2012 and will increase by 5% each year. Calculate

(i) the value of the grant in 2022, [2]

(ii) the total amount the college will receive in the years 2012 to 2022 inclusive. [2]

Answers: **(a) (i)** 3; **(ii)** 57; **(iii)** 570; **(b) (i)** \$6520; **(ii)** \$56,820 **N11/12/Q10**

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20 **(a)** In an arithmetic progression, the sum of the first n terms, denoted by S_n, is given by

$$S_n = n^2 + 8n.$$

Find the first term and the common difference. [3]

(b) In a geometric progression, the second term is 9 less than the first term. The sum of the second and third terms is 30. Given that all the terms of the progression are positive, find the first term. [5]

Answers: **(a)** $a = 9$, $d = 2$ **(b)** $a = 27$ **J12/12/Q7**

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21 **(a)** The first and last terms of an arithmetic progression are 12 and 48 respectively. The sum of the first four terms is 57. Find the number of terms in the progression. [4]

(b) The third term of a geometric progression is four times the first term. The sum of the first six terms is k times the first term. Find the possible values of k. [4]

Answers: **(a)** $n = 25$. **(b)** $k = 63$ or -21. **J13/12/Q10**

---

22 The 1st, 2nd and 3rd terms of a geometric progression are the 1st, 9th and 21st terms respectively of an arithmetic progression. The 1st term of each progression is 8 and the common ratio of the geometric progression is r, where $r \neq 1$. Find

(i) the value of r, [4]

(ii) the 4th term of each progression. [3]

Answers: **(i)** $r = 1.5$ **(ii)** 27 and 9.5 **J14/12/Q6**

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23 **(a)** The first, second and last terms in an arithmetic progression are 56, 53 and -22 respectively. Find the sum of all the terms in the progression. [4]

(b) The first, second and third terms of a geometric progression are $2k + 6$, $2k$ and $k + 2$ respectively, where k is a positive constant.

(i) Find the value of k. [3]

(ii) Find the sum to infinity of the progression. [2]

Answer: **(a)** 459; **(bi)** $k = 6$, **(ii)** 54. **J15/12/Q8**

24 (a) In a geometric progression, all the terms are positive, the second term is 24 and the fourth term is $13\frac{1}{2}$. Find

(i) the first term, [3]

(ii) the sum to infinity of the progression. [2]

(b) A circle is divided into n sectors in such a way that the angles of the sectors are in arithmetic progression. The smallest two angles are 3° and 5°. Find the value of n. [4]

Answers: **(a)** **(i)** 32; **(ii)** 128; **(b)** n = 18. **N12/12/Q8**

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25 (a) An athlete runs the first mile of a marathon in 5 minutes. His speed reduces in such a way that each mile takes 12 seconds longer than the preceding mile.

(i) Given that the nth mile takes 9 minutes, find the value of n. [2]

(ii) Assuming that the length of the marathon is 26 miles, find the total time, in hours and minutes, to complete the marathon. [2]

(b) The second and third terms of a geometric progression are 48 and 32 respectively. Find the sum to infinity of the progression. [4]

Answers: **(a)** **(i)** 21, **(ii)** 3 hours 15 minutes. **(b)** 216. **N13/12/Q7**

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26 (a) The sum, S_n, of the first n terms of an arithmetic progression is given by $S_n = 32n - n^2$. Find the first term and the common difference. [3]

(b) A geometric progression in which all the terms are positive has sum to infinity 20. The sum of the first two terms is 12.8. Find the first term of the progression. [5]

Answers: **(a)** a = 31, d = –2, **(b)** a = 8 **N14/12/Q8**

Homework: Arithmetic and Geometric Progression – Variants 11 & 13

1 **(a)** The first term of an arithmetic progression is -2222 and the common difference is 17. Find the value of the first positive term. [3]

(b) The first term of a geometric progression is $\sqrt{3}$ and the second term is $2\cos\theta$, where $0 < \theta < \pi$. Find the set of values of θ for which the progression is convergent. [5]

Answers: (a) 5; (b) $\frac{\pi}{6} < \theta < \frac{5\pi}{6}$. **13/J15/9**

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2 **(a)** The third and fourth terms of a geometric progression are $\frac{1}{3}$ and $\frac{2}{9}$ respectively. Find the sum to infinity of the progression. [4]

(b) A circle is divided into 5 sectors in such a way that the angles of the sectors are in arithmetic progression. Given that the angle of the largest sector is 4 times the angle of the smallest sector, find the angle of the largest sector. [4]

Answers: (a) $2\frac{1}{4}$ (b) 115.2° or 2.01 **11/J15/7**

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3 Three geometric progressions, P, Q and R, are such that their sums to infinity are the first three terms respectively of an arithmetic progression.

Progression P is $2, 1, \frac{1}{2}, \frac{1}{4}, \ldots$.

Progression Q is $3, 1, \frac{1}{3}, \frac{1}{9}, \ldots$.

(i) Find the sum to infinity of progression R. [3]

(ii) Given that the first term of R is 4, find the sum of the first three terms of R. [3]

Answers: (i) 5; (ii) 4.96 **13/N14/4**

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4 **(i)** A geometric progression has first term a ($a \neq 0$), common ratio r and sum to infinity S. A second geometric progression has first term a, common ratio $2r$ and sum to infinity $3S$. Find the value of r. [3]

(ii) An arithmetic progression has first term 7. The nth term is 84 and the $(3n)$th term is 245. Find the value of n. [4]

Answers: (i) $\frac{2}{5}$ (ii) 23 **11/N14/7**

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5 The first term in a progression is 36 and the second term is 32.

(i) Given that the progression is geometric, find the sum to infinity. [2]

(ii) Given instead that the progression is arithmetic, find the number of terms in the progression if the sum of all the terms is 0. [3]

Answers: (i) 324; (ii) 19. 13/J14/2

6 An arithmetic progression has first term a and common difference d. It is given that the sum of the first 200 terms is 4 times the sum of the first 100 terms.

(i) Find d in terms of a. [3]

(ii) Find the 100th term in terms of a. [2]

Answers: (i) $d = 2a$; (ii) $199a$. 11/J14/5

7 (a) In a geometric progression, the sum to infinity is equal to eight times the first term. Find the common ratio. [2]

(b) In an arithmetic progression, the fifth term is 197 and the sum of the first ten terms is 2040. Find the common difference. [4]

Answer: $\frac{7}{8}$, 14 13/N13/5

8 (a) In an arithmetic progression the sum of the first ten terms is 400 and the sum of the next ten terms is 1000. Find the common difference and the first term. [5]

(b) A geometric progression has first term a, common ratio r and sum to infinity 6. A second geometric progression has first term $2a$, common ratio r^2 and sum to infinity 7. Find the values of a and r. [5]

Answers: (a) $d = 6$, $a = 13$; (b) $r = 5/7$ (or 0.714), $a = 12/7$ (or 1.71). 11/N13/9

9 (a) In an arithmetic progression, the sum, S_n, of the first n terms is given by $S_n = 2n^2 + 8n$. Find the first term and the common difference of the progression. [3]

(b) The first 2 terms of a geometric progression are 64 and 48 respectively. The first 3 terms of the geometric progression are also the 1st term, the 9th term and the nth term respectively of an arithmetic progression. Find the value of n. [5]

Answers: (a) $a = 10$, $d = 4$; (b) $n = 15$. 13/J13/9

10 The third term of a geometric progression is -108 and the sixth term is 32. Find

(i) the common ratio, [3]

(ii) the first term, [1]

(iii) the sum to infinity. [2]

Answers: (i) $-2/3$; (ii) -243; (iii) -145.8. 11/J13/4

11 The first term of a geometric progression is $5\frac{1}{3}$ and the fourth term is $2\frac{1}{4}$. Find

(i) the common ratio, [3]

(ii) the sum to infinity. [2]

Answers: (i) 0.75 or $\frac{3}{4}$; (ii) $21\frac{1}{3}$. 13/N12/5

12 The first term of an arithmetic progression is 61 and the second term is 57. The sum of the first n terms is n. Find the value of the positive integer n. [4]

Answer: 31. 11/N12/1

13 The first term of an arithmetic progression is 12 and the sum of the first 9 terms is 135.

(i) Find the common difference of the progression. [2]

The first term, the ninth term and the nth term of this arithmetic progression are the first term, the second term and the third term respectively of a geometric progression.

(ii) Find the common ratio of the geometric progression and the value of n. [5]

Answers: (i) $d = ¾$ (ii) $r = 1½$, $n = 21$ 13/J12/6

14 **(a)** The first two terms of an arithmetic progression are 1 and $\cos^2 x$ respectively. Show that the sum of the first ten terms can be expressed in the form $a - b\sin^2 x$, where a and b are constants to be found. [3]

(b) The first two terms of a geometric progression are 1 and $\frac{1}{3}\tan^2\theta$ respectively, where $0 < \theta < \frac{1}{2}\pi$.

(i) Find the set of values of θ for which the progression is convergent. [2]

(ii) Find the exact value of the sum to infinity when $\theta = \frac{1}{6}\pi$. [2]

Answers: (i) $10 - 45\sin^2 x$ (b)(i) $0 < \theta < \frac{1}{3}\pi$ (ii) $\frac{9}{8}$ 11/J12/7

15	The first and second terms of a progression are 4 and 8 respectively. Find the sum of the first 10 terms given that the progression is **(i)** an arithmetic progression, [2] **(ii)** a geometric progression. [2]	
	Answers: (i) 220, (ii) 4092	**13/N11/2**
16	**(a)** The sixth term of an arithmetic progression is 23 and the sum of the first ten terms is 200. Find the seventh term. [4] **(b)** A geometric progression has first term 1 and common ratio r. A second geometric progression has first term 4 and common ratio $\frac{1}{4}r$. The two progressions have the same sum to infinity, S. Find the values of r and S. [3]	
	Answers: (a) 29; (b) $r = 4/5$, $S = 5$.	**11/N11/6**
17	**(a)** A geometric progression has a third term of 20 and a sum to infinity which is three times the first term. Find the first term. [4] **(b)** An arithmetic progression is such that the eighth term is three times the third term. Show that the sum of the first eight terms is four times the sum of the first four terms. [4]	
	Answer: (a) 45.	**13/J11/6**
18	A television quiz show takes place every day. On day 1 the prize money is \$1000. If this is not won the prize money is increased for day 2. The prize money is increased in a similar way every day until it is won. The television company considered the following two different models for increasing the prize money. Model 1: Increase the prize money by \$1000 each day. Model 2: Increase the prize money by 10% each day. On each day that the prize money is not won the television company makes a donation to charity. The amount donated is 5% of the value of the prize on that day. After 40 days the prize money has still not been won. Calculate the total amount donated to charity **(i)** if Model 1 is used, [4] **(ii)** if Model 2 is used. [3]	
	Answers: (i) \$41 000; (ii) \$22 100.	**11/J11/8**
19	**(a)** A geometric progression has first term 100 and sum to infinity 2000. Find the second term. [3] **(b)** An arithmetic progression has third term 90 and fifth term 80. **(i)** Find the first term and the common difference. [2]	

(ii) Find the value of m given that the sum of the first m terms is equal to the sum of the first $(m+1)$ terms. [2]

(iii) Find the value of n given that the sum of the first n terms is zero. [2]

13/N10/9

Answers: (a) 95; (b)(i) 100, –5; (ii) 20; (iii) 41.

20 (a) The fifth term of an arithmetic progression is 18 and the sum of the first 5 terms is 75. Find the first term and the common difference. [4]

(b) The first term of a geometric progression is 16 and the fourth term is $\frac{27}{4}$. Find the sum to infinity of the progression. [3]

Answers: (a) 12, 1.5; (b) 64.

11/N10/6

21 The first term of a geometric progression is 12 and the second term is –6. Find

(i) the tenth term of the progression, [3]

(ii) the sum to infinity. [2]

13/J10/1

Answers: (i) $-\frac{3}{128}$; (ii) 8.

22 The ninth term of an arithmetic progression is 22 and the sum of the first 4 terms is 49.

(i) Find the first term of the progression and the common difference. [4]

The nth term of the progression is 46.

(ii) Find the value of n. [2]

Answers: (i) 10, 1.5; (ii) 25.

11/J10/3

23 The first term of a progression is $4x$ and the second term is x^2.

(i) For the case where the progression is arithmetic with a common difference of 12, find the possible values of x and the corresponding values of the third term. [4]

(ii) For the case where the progression is geometric with a sum to infinity of 8, find the third term. [4]

N15/11/Q8

24 A ball is such that when it is dropped from a height of 1 metre it bounces vertically from the ground to a height of 0.96 metres. It continues to bounce on the ground and each time the height the ball reaches is reduced. Two different models, A and B, describe this.

Model A : The height reached is reduced by 0.04 metres each time the ball bounces.

Model B : The height reached is reduced by 4% each time the ball bounces.

(i) Find the total distance travelled vertically (up and down) by the ball from the 1st time it hits the ground until it hits the ground for the 21st time,

(a) using model A, [3]

(b) using model B. [3]

(ii) Show that, under model B, even if there is no limit to the number of times the ball bounces, the total vertical distance travelled after the first time it hits the ground cannot exceed 48 metres. [2]

N15/13/Q6

25 (a) The first term of a geometric progression in which all the terms are positive is 50. The third term is 32. Find the sum to infinity of the progression. [3]

(b) The first three terms of an arithmetic progression are $2\sin x$, $3\cos x$ and $(\sin x + 2\cos x)$ respectively, where x is an acute angle.

(i) Show that $\tan x = \frac{4}{3}$. [3]

(ii) Find the sum of the first twenty terms of the progression. [3]

Answers: **(a)** 250 **(b)(ii)** 70 J16/11/Q9

26 The 1st, 3rd and 13th terms of an arithmetic progression are also the 1st, 2nd and 3rd terms respectively of a geometric progression. The first term of each progression is 3. Find the common difference of the arithmetic progression and the common ratio of the geometric progression. [5]

Answer: $r = 5$, $d = 6$. J16/13/Q4

27 (a) An arithmetic progression has a first term of 32, a 5th term of 22 and a last term of –28. Find the sum of all the terms in the progression. [4]

(b) Each year a school allocates a sum of money for the library. The amount allocated each year increases by 2.5% of the amount allocated the previous year. In 2005 the school allocated \$2000. Find the total amount allocated in the years 2005 to 2014 inclusive. [3]

Answers: **(a)** 50 **(b)** £22400 J17/11/Q4

28 The common ratio of a geometric progression is r. The first term of the progression is $(r^2 - 3r + 2)$ and the sum to infinity is S.

(i) Show that $S = 2 - r$. [2]

(ii) Find the set of possible values that S can take. [2]

Answer: **(ii)** $1 < S < 3$. **J17/13/Q2**

29 (a) A geometric progression has a second term of 12 and a sum to infinity of 54. Find the possible values of the first term of the progression. [4]

(b) The nth term of a progression is $p + qn$, where p and q are constants, and S_n is the sum of the first n terms.

(i) Find an expression, in terms of p, q and n, for S_n. [3]

(ii) Given that $S_4 = 40$ and $S_6 = 72$, find the values of p and q. [2]

Answers: **(a)** 18 and 36 **(b)(i)** $\frac{n}{2}(2p + nq + q)$ or equivalent **(b)(ii)** $p = 5, q = 2$ **J18/11/Q8**

30 The common ratio of a geometric progression is 0.99. Express the sum of the first 100 terms as a percentage of the sum to infinity, giving your answer correct to 2 significant figures. [5]

Answer: 63%. **J18/13/Q3**

31 The sum of the 1st and 2nd terms of a geometric progression is 50 and the sum of the 2nd and 3rd terms is 30. Find the sum to infinity. [6]

Answer: 625/8 **N16/11/Q5**

32 (a) Two convergent geometric progressions, P and Q, have the same sum to infinity. The first and second terms of P are 6 and $6r$ respectively. The first and second terms of Q are 12 and $-12r$ respectively. Find the value of the common sum to infinity. [3]

(b) The first term of an arithmetic progression is $\cos\theta$ and the second term is $\cos\theta + \sin^2\theta$, where $0 \leqslant \theta \leqslant \pi$. The sum of the first 13 terms is 52. Find the possible values of θ. [5]

Answers: **(a)** 9; **(b)** 0.841, 2.09. **N16/13/Q9**

33 (a) A geometric progression has first term $3a$ and common ratio r. A second geometric progression has first term a and common ratio $-2r$. The two progressions have the same sum to infinity. Find the value of r. [3]

(b) The first two terms of an arithmetic progression are 15 and 19 respectively. The first two terms of a second arithmetic progression are 420 and 415 respectively. The two progressions have the same sum of the first n terms. Find the value of n. [3]

Answers: **(a)** – 2 / 7 **(b)** 91 N17/11/Q3

34 An arithmetic progression has first term -12 and common difference 6. The sum of the first n terms exceeds 3000. Calculate the least possible value of n. [4]

Answer: 35 N17/13/Q1

35 The first term of a series is 6 and the second term is 2.

(i) For the case where the series is an arithmetic progression, find the sum of the first 80 terms. [3]

(ii) For the case where the series is a geometric progression, find the sum to infinity. [2]

Answers: **(i)** –12160, **(ii)** 9 N18/11/Q4

36 In an arithmetic progression the first term is a and the common difference is 3. The nth term is 94 and the sum of the first n terms is 1420. Find n and a. [6]

Answer: $n = 40$ $a = -23$ N18/13/Q5

Differentiation

Example 1

Write the following in the form ax^n.

a) $\frac{2}{x^3}$ b) $4\sqrt{x}$ c) $\frac{2}{\sqrt{x}}$

d) $\frac{3}{\sqrt[3]{x}}$ e) $\sqrt{x^5}$ f) $\frac{1}{\sqrt{x^3}}$

Finding the gradient of the curve

Figure 5.3 shows the part of the graph $y = x^2$ which lies between $x = -1$ and $x = 3$. What is the value of the gradient at the point P(3, 9)?

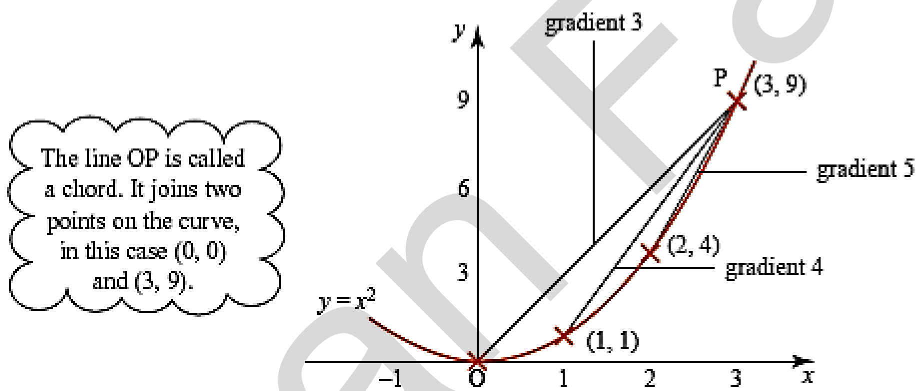


Figure 5.3

You have already seen that drawing the tangent at the point by hand provides only an approximate answer. A different approach is to calculate the gradients of chords to the curve. These will also give only approximate answers for the gradient of the curve, but they will be based entirely on calculation and not depend on your drawing skill. Three chords are marked on figure 5.3.

Chord (0, 0) to (3, 9): gradient $= \frac{9-0}{3-0} = 3$

Chord (1, 1) to (3, 9): gradient $= \frac{9-1}{3-1} = 4$

Chord (2, 4) to (3, 9): gradient $= \frac{9-4}{3-2} = 5$

Clearly none of these three answers is exact, but which of them is the most accurate?

Of the three chords, the one closest to being a tangent is that joining (2, 4) to (3, 9), the two points that are closest together.

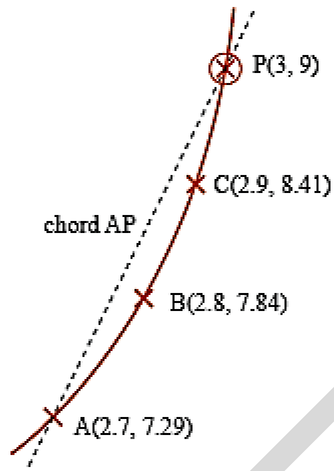


Figure 5.4

The x co-ordinate of point A is 2.7, the y co-ordinate 2.7^2, or 7.29 (since the point lies on the curve $y = x^2$). Similarly B and C are (2.8, 7.84) and (2.9, 8.41). The gradients of the chords joining each point to (3, 9) are as follows.

Chord (2.7, 7.29) to (3, 9): gradient $= \dfrac{9 - 7.29}{3 - 2.7} = 5.7$

Chord (2.8, 7.84) to (3, 9): gradient $= \dfrac{9 - 7.84}{3 - 2.8} = 5.8$

Chord (2.9, 8.41) to (3, 9): gradient $= \dfrac{9 - 8.41}{3 - 2.9} = 5.9$

These results are getting closer to the gradient of the tangent. What happens if you take points much closer to (3, 9), for example (2.99, 8.9401) and (2.999, 8.994001)?

The gradients of the chords joining these to (3, 9) work out to be 5.99 and 5.999 respectively.

It looks as if the gradients are approaching the value 6, and if so this is the gradient of the tangent at (3, 9).

Taking this method to its logical conclusion, you might try to calculate the gradient of the 'chord' from (3, 9) to (3, 9), but this is undefined because there is a zero in the denominator. So although you can find the gradient of a chord which is as close as you like to the tangent, it can never be exactly that of the tangent.

What you need is a way of making that final step from a chord to a tangent.

The concept of a *limit* enables us to do this, as you will see in the next section. It allows us to confirm that in the limit as point Q tends to point P(3, 9), the chord QP tends to the tangent of the curve at P, and the gradient of QP tends to 6 (see figure 5.5).

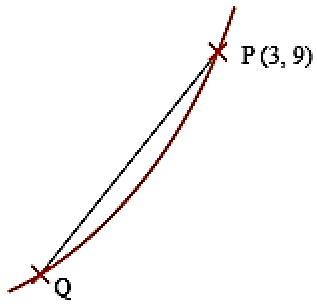


Figure 5.5

Finding the gradient from first principles

Although the work in the previous section was more formal than the method of drawing a tangent and measuring its gradient, it was still somewhat experimental. The result that the gradient of $y = x^2$ at (3, 9) is 6 was a sensible conclusion, rather than a proved fact.

In this section the method is formalised and extended.

Take the point P(3, 9) and another point Q close to (3, 9) on the curve $y = x^2$. Let the x co-ordinate of Q be $3 + h$ where h is small. Since $y = x^2$ at Q, the y co-ordinate of Q will be $(3 + h)^2$.

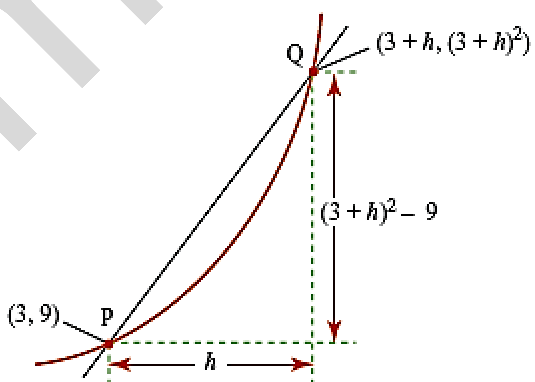


Figure 5.6

From figure 5.6, the gradient of PQ is $\frac{(3+h)^2-9}{h}$

$$= \frac{9+6h+h^2-9}{h}$$

$$= \frac{6h+h^2}{h}$$

$$= \frac{h(6+h)}{h}$$

$$= 6+h.$$

For example, when $h=0.001$, the gradient of PQ is 6.001, and when $h=-0.001$, the gradient of PQ is 5.999. The gradient of the tangent at P is between these two values. Similarly the gradient of the tangent would be between $6-h$ and $6+h$ for all small non-zero values of h.

For this to be true the gradient of the tangent at (3, 9) must be *exactly* 6.

The definition of $f'(x)$ is given as $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$.

Example 2

Find an expression for $f'(x)$ when $f(x) = x^3$.

If $y = x^n$ then $\frac{dy}{dx} = nx^{n-1}$

If $y = ax^n$ then $\frac{dy}{dx} = anx^{n-1}$

Example 3

Find the derived function in each case.

a) $f(x) = x^8$

b) $f(x) = \frac{6}{x^4}$

c) $f(x) = 3x$

d) $f(x) = 5$

Example 4

Find $\frac{dy}{dx}$ when $y = 3\sqrt{x}$.

Example 5

Find $\frac{dy}{dx}$ when

a) $y = x^5 + 4x^4$ **b)** $y = (2x - 5)^2$ **c)** $y = \frac{3x^6 + 8}{x^2}$.

Example 6

Differentiate $f(x) = \frac{(x^2 + 1)(x - 5)}{x}$

Example 7

Find $\frac{d}{dx}\left(24\sqrt[3]{x} - \frac{1}{2\sqrt{x}}\right)$.

Example 8

Find $\frac{dy}{dx}$ when $y = \frac{2}{(3x^2 + 1)^4}$.

Example 9

Given that $y = \sqrt{x} - \frac{8}{x^2}$, find

(i) $\frac{dy}{dx}$

(ii) the gradient of the curve at the point $\left(4, 1\frac{1}{2}\right)$.

Example 10

Find the points on the curve with equation $y = x^3 + 6x^2 + 5$ where the value of the gradient is -9.

Example 11

Find the gradient of the tangent to the curve $y = 4x^3 - 2x^2 + 1$ at the point (1, 3).

Example 12

Find the coordinates of the point on the curve with equation $y = 5 - 3x - 4x^2$ where the gradient is 1.

Example 13

Find the gradient of the tangent to the curve $y = 4x^2 - 5x + 2$ at each of the points where the curve meets the line $y = 7x - 6$.

Example 14

Find the equation of the tangent to the curve $y = 3x^3 - 6x^2 + x$ at the point (1, −2).

Example 15

Find the equation of the normal to the curve $y = 4\sqrt{x} + 7$ at the point where $x = 16$.

Example 16

The equation of a curve is $y = 4x - x^2$. The normal to the curve at the point $P(1, 3)$. meets the curve again at Q. Find the coordinates of Q.

Example 17

A curve has equation $y = \frac{16}{x} - 4\sqrt{x}$. The normal to the curve at the point (4, −4) meets the y axis at the point P. Find the co-ordinates of P.

Increasing and Decreasing Functions

When the gradient of a function is positive we say the function is an **increasing function.**
A function $f(x)$ is increasing for $a < x < b$ if $f'(x) > 0$ for $a < x < b$.

When the gradient of a function is negative we say the function is a **decreasing function.**
A function $f(x)$ is decreasing for $a < x < b$ if $f'(x) < 0$ for $a < x < b$.

Example 18

Show that $y = x^3 + x$ is an increasing function.

Example 19

Find the range of values of x for which the function $y = x^2 - 6x$ is a decreasing function.

Example 20

The equation of a curve is $y = \frac{1}{6}(2x - 3)^3 - 4x$.

(i) Find $\frac{dy}{dx}$.

(ii) Find the equation of the tangent to the curve at the point where the curve intersects the y axis.

(iii) Find the set of values of x for which $\frac{1}{6}(2x - 3)^3 - 4x$ is an increasing function of x.

[Cambridge AS & A Level Mathematics 9709, Paper 12 Q10 June 2010]

Example 21

The equation of a curve is $y = x^2 - 3x + 4$.

(i) Show that the whole of the curve lies above the x axis.

(ii) Find the set of values of x for which $x^2 - 3x + 4$ is a decreasing function of x.

The equation of a line is $y + 2x = k$, where k is a constant.

(iii) In the case where $k = 6$, find the co-ordinates of the points of intersection of the line and the curve.

(iv) Find the value of k for which the line is a tangent to the curve.

[Cambridge AS & A Level Mathematics 9709, Paper 1 Q10 June 2005]

The second derivative

Example 22

Given that $y = x^5 + 2x$, find $\frac{d^2y}{dx^2}$.

Example 23

Find $\frac{d^2y}{dx^2}$ when $y = (4x^2 + 1)^2$.

Example 24

Find $\frac{d^2y}{dx^2}$ when $\frac{dy}{dx} = \frac{3}{\sqrt{x}}$.

Example 25

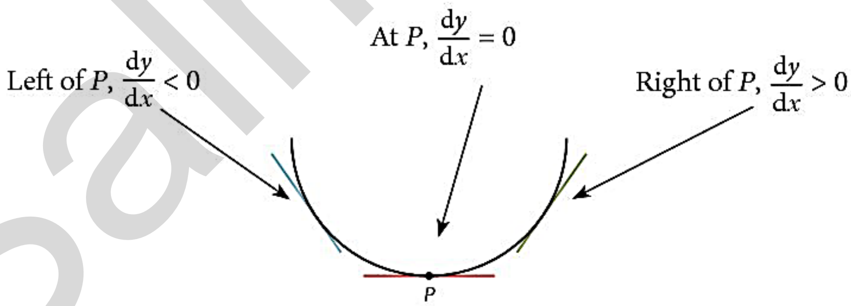
Given that $f(x) = 8x^3 - 3x + \frac{4}{x}$, find the value of $f''(-2)$.

Stationary Points

Any point on a curve where $\frac{dy}{dx} = 0$ is called a **stationary point**.

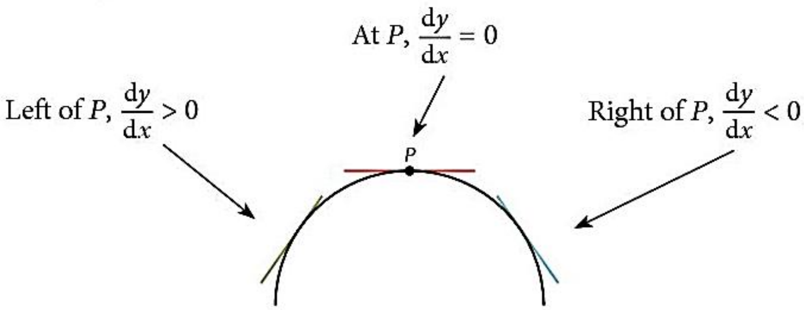
There are three types of stationary points: **minimum** point, **maximum** point, and **point of inflexion**.

Minimum point



Note: At a minimum point, the sign of $\frac{dy}{dx}$ changes from $-$ to $+$ so the gradient of $\frac{dy}{dx}$ is $+$, i.e. $\frac{d^2y}{dx^2} \geq 0$.

Maximum point



Note: At a maximum point $\frac{d^2y}{dx^2} \leq 0$.

Stationary points which are either a maximum point or a minimum point are often called turning points.

Example 26

Find the coordinates of the stationary points of the curve $y = 6x - 2x^3$ and **determine** their nature. **Sketch** the curve.

Example 27

Find the coordinates of the stationary points of the curve $y = \frac{2}{2x-1} + x$ and determine their nature.

Example 28

Given that $y = 2x^3 + 3x^2 - 12x$

(i) find $\frac{dy}{dx}$, and find the values of x for which $\frac{dy}{dx} = 0$

(ii) find the value of $\frac{d^2y}{dx^2}$ at each stationary point and hence determine its nature

(iii) find the y values of each of the stationary points

(iv) sketch the curve given by $y = 2x^3 + 3x^2 - 12x$.

Example 29

The equation of a curve C is $y = 2x^2 - 8x + 9$ and the equation of a line L is $x + y = 3$.

(i) Find the x co-ordinates of the points of intersection of L and C.

(ii) Show that one of these points is also the stationary point of C.

[Cambridge AS & A Level Mathematics 9709, Paper 1 Q4 June 2008]

Example 30

November 2016/11 Question 11

The point $P(3, 5)$ lies on the curve $y = \frac{1}{x-1} - \frac{9}{x-5}$.

(i) Find the x-coordinate of the point where the normal to the curve at P intersects the x-axis. [5]

(ii) Find the x-coordinate of each of the stationary points on the curve and determine the nature of each stationary point, justifying your answers. [6]

Example 31

November 2016/13 Question 11

A curve has equation $y = (kx - 3)^{-1} + (kx - 3)$, where k is a non-zero constant.

(i) Find the x-coordinates of the stationary points in terms of k, and determine the nature of each stationary point, justifying your answers. [7]

Example 32

June 2016/13 Question 5

A curve has equation $y = 8x + (2x - 1)^{-1}$. Find the values of x at which the curve has a stationary point and determine the nature of each stationary point, justifying your answers. [7]

Example 33

November 2015/11 Question 5

A curve has equation $y = \frac{8}{x} + 2x$.

(i) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. [3]

(ii) Find the coordinates of the stationary points and state, with a reason, the nature of each stationary point. [5]

Example 34

A stone is projected vertically upwards with a speed of $30\,m\,s^{-1}$.
Its height, h m, above the ground after t seconds ($t < 6$) is given by:

$$h = 30t - 5t^2.$$

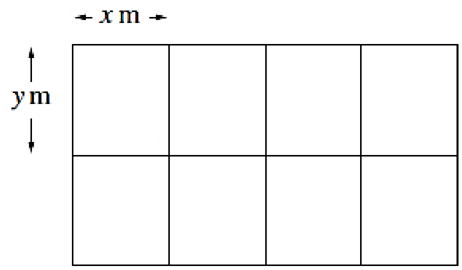
(I) Find $\frac{dh}{dt}$ and $\frac{d^2h}{dt^2}$.

(II) Find the maximum height reached.

(III) Sketch the graph of h against t.

Example 35

June 2016/11 Question 5



A farmer divides a rectangular piece of land into 8 equal-sized rectangular sheep pens as shown in the diagram. Each sheep pen measures x m by y m and is fully enclosed by metal fencing. The farmer uses 480 m of fencing.

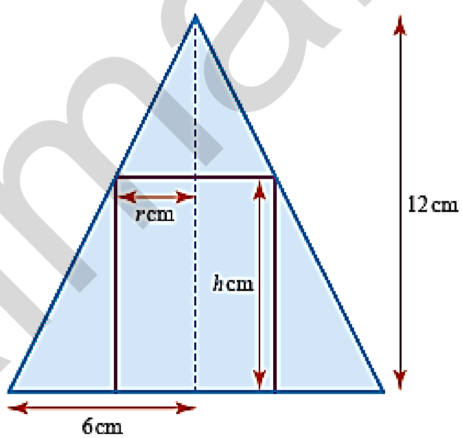
(i) Show that the total area of land used for the sheep pens, A m^2, is given by

$$A = 384x - 9.6x^2. \quad [3]$$

(ii) Given that x and y can vary, find the dimensions of each sheep pen for which the value of A is a maximum. (There is no need to verify that the value of A is a maximum.) [3]

Example 36

The diagram shows the cross-section of a hollow cone and a circular cylinder. The cone has radius 6 cm and height 12 cm, and the cylinder has radius r cm and height h cm. The cylinder just fits inside the cone with all of its upper edge touching the surface of the cone.



(i) Express h in terms of r and hence show that the volume, V cm^3, of the cylinder is given by

$$V = 12\pi r^2 - 2\pi r^3$$

(ii) Given that r varies, find the stationary value of V.

[Cambridge AS & A Level Mathematics 9709, Paper 1 Q5 November 2005]

Example 37

The radius r cm of a circular ripple made by dropping a stone into a pond is increasing at a rate of $8\ \text{cm s}^{-1}$. At what rate is the area $A\ \text{cm}^2$ enclosed by the ripple increasing when the radius is 25 cm?

Example 38

The radius, r, of a circle is increasing at the rate of $\frac{2}{r^2}\ \text{m s}^{-1}$.

Find the rate at which the area, A, is increasing when $r = 8$.

Example 39

November 2016/12 Question 7

The equation of a curve is $y = 2 + \frac{3}{2x - 1}$.

(i) Obtain an expression for $\frac{dy}{dx}$. [2]

(ii) Explain why the curve has no stationary points. [1]

At the point P on the curve, $x = 2$.

(iii) Show that the normal to the curve at P passes through the origin. [4]

(iv) A point moves along the curve in such a way that its x-coordinate is decreasing at a constant rate of 0.06 units per second. Find the rate of change of the y-coordinate as the point passes through P. [2]

Example 40

June 2016/13 Question 7

The point $P(x, y)$ is moving along the curve $y = x^2 - \frac{10}{3}x^{\frac{3}{2}} + 5x$ in such a way that the rate of change of y is constant. Find the values of x at the points at which the rate of change of x is equal to half the rate of change of y. [7]

Example 41

November 2015/12 Question 3

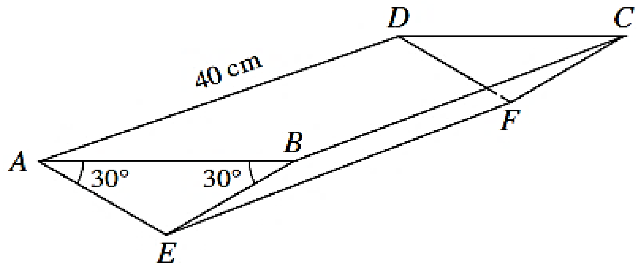


Fig. 1

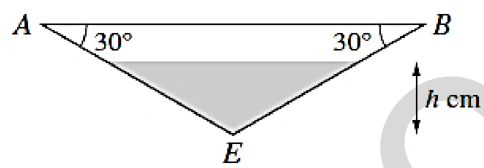


Fig. 2

Fig. 1 shows an open tank in the shape of a triangular prism. The vertical ends ABE and DCF are identical isosceles triangles. Angle ABE = angle $BAE = 30°$. The length of AD is 40 cm. The tank is fixed in position with the open top $ABCD$ horizontal. Water is poured into the tank at a constant rate of $200\,\text{cm}^3\,\text{s}^{-1}$. The depth of water, t seconds after filling starts, is h cm (see Fig. 2).

(i) Show that, when the depth of water in the tank is h cm, the volume, $V\,\text{cm}^3$, of water in the tank is given by $V = (40\sqrt{3})h^2$. [3]

(ii) Find the rate at which h is increasing when $h = 5$. [3]

Homework: Differentiation Variant 12

1 A curve has equation $y = x^3 + 3x^2 - 9x + k$, where k is a constant.

(i) Write down an expression for $\frac{dy}{dx}$. [2]

(ii) Find the x-coordinates of the two stationary points on the curve. [2]

(iii) Hence find the two values of k for which the curve has a stationary point on the x-axis. [3]

Answers: (i) $3x^2 + 6x - 9$; (ii) $x = -3$ or 1; (iii) $k = -27$ or 5. **N02/Q8**

---

2 (a) Differentiate $4x + \frac{6}{x^2}$ with respect to x. [2]

Answers: (a) $4 - \frac{12}{x^3}$; (b) $2x^2 - \frac{6}{x} + c$. **J03/Q3**

---

3 A solid rectangular block has a base which measures $2x$ cm by x cm. The height of the block is y cm and the volume of the block is 72 cm^3.

(i) Express y in terms of x and show that the total surface area, $A\text{ cm}^2$, of the block is given by

$$A = 4x^2 + \frac{216}{x}.$$ [3]

Given that x can vary,

(ii) find the value of x for which A has a stationary value, [3]

(iii) find this stationary value and determine whether it is a maximum or a minimum. [3]

Answers: (i) $y = \frac{36}{x^2}$; (ii) $x = 3$; (iii) $A = 108\text{ cm}^2$, minimum. **N03/Q8**

---

4 A curve is such that $\frac{dy}{dx} = \frac{6}{\sqrt{(4x - 3)}}$ and $P(3, 3)$ is a point on the curve.

(i) Find the equation of the normal to the curve at P, giving your answer in the form $ax + by = c$. [3]

Answers: (i) $x + 2y = 9$; (ii) $y = 3\sqrt{4x - 3} - 6$. **N04/Q7**

5 A curve has equation $y = x^2 + \frac{2}{x}$.

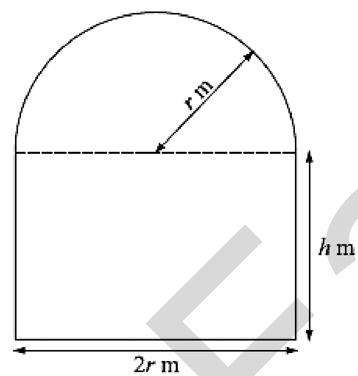
(i) Write down expressions for $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. [3]

(ii) Find the coordinates of the stationary point on the curve and determine its nature. [4]

N04/Q10

Answers: (i) $2x - \frac{2}{x^2}$, $2 + \frac{4}{x^3}$ (ii) (1, 3), Minimum point; (iii) 14.2π or 44.6.

6



The diagram shows a glass window consisting of a rectangle of height h m and width $2r$ m and a semicircle of radius r m. The perimeter of the window is 8 m.

(i) Express h in terms of r. [2]

(ii) Show that the area of the window, A m^2, is given by

$$A = 8r - 2r^2 - \tfrac{1}{2}\pi r^2.$$ [2]

Given that r can vary,

(iii) find the value of r for which A has a stationary value, [4]

(iv) determine whether this stationary value is a maximum or a minimum. [2]

J04/Q8

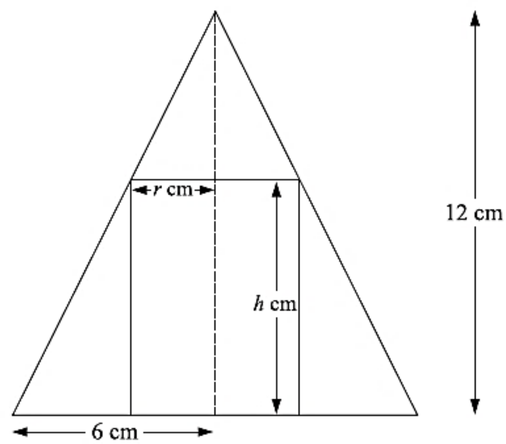
Answers: (i) $h = 4 - r - \frac{1}{2}\pi r$; (iii) 1.12 or $\frac{8}{4+\pi}$; (iv) maximum.

7 Find the gradient of the curve $y = \frac{12}{x^2 - 4x}$ at the point where $x = 3$. [4]

J05/Q2

Answer: $-\frac{8}{3}$.

8



The diagram shows the cross-section of a hollow cone and a circular cylinder. The cone has radius 6 cm and height 12 cm, and the cylinder has radius r cm and height h cm. The cylinder just fits inside the cone with all of its upper edge touching the surface of the cone.

(i) Express h in terms of r and hence show that the volume, V cm^3, of the cylinder is given by

$$V = 12\pi r^2 - 2\pi r^3.$$ [3]

(ii) Given that r varies, find the stationary value of V. [4]

Answers: (i) $h = 12 - 2r$; (ii) 64π or 201 cm^3. **N05/Q5**

9 A function f is defined by $f : x \mapsto (2x - 3)^3 - 8$, for $2 \leqslant x \leqslant 4$.

(i) Find an expression, in terms of x, for $f'(x)$ and show that f is an increasing function. [4]

(ii) Find an expression, in terms of x, for $f^{-1}(x)$ and find the domain of f^{-1}. [4]

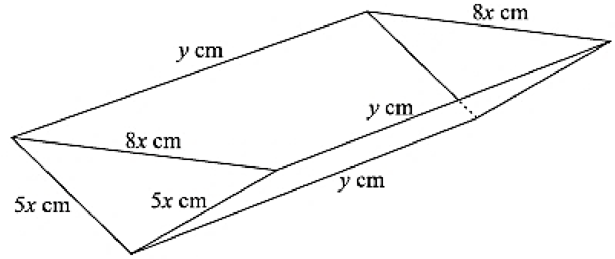
N05/Q8

Answers: (i) $6(2x-3)^2$; (ii) $\frac{\sqrt[3]{(x+8}+3}{2}$, $-7 \leqslant x \leqslant 117$.

10 A curve has equation $y = \frac{k}{x}$. Given that the gradient of the curve is -3 when $x = 2$, find the value of the constant k. [3]

Answer: 12. **J06/Q1**

11



The diagram shows an open container constructed out of 200 cm^2 of cardboard. The two vertical end pieces are isosceles triangles with sides $5x$ cm, $5x$ cm and $8x$ cm, and the two side pieces are rectangles of length y cm and width $5x$ cm, as shown. The open top is a horizontal rectangle.

(i) Show that $y = \frac{200 - 24x^2}{10x}$. [3]

(ii) Show that the volume, V cm^3, of the container is given by $V = 240x - 28.8x^3$. [2]

Given that x can vary,

(iii) find the value of x for which V has a stationary value, [3]

(iv) determine whether it is a maximum or a minimum stationary value. [2]

Answer: (iii) $1\frac{2}{3}$; (iv) Maximum. **N06/Q9**

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12 The equation of a curve is $y = 2x + \frac{8}{x^2}$.

(i) Obtain expressions for $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. [3]

(ii) Find the coordinates of the stationary point on the curve and determine the nature of the stationary point. [3]

(iii) Show that the normal to the curve at the point $(-2, -2)$ intersects the x-axis at the point $(-10, 0)$. [3]

Answers: (i) $2 - \frac{16}{x^3}$, $\frac{48}{x^4}$; (ii) (2, 6), Minimum; (iv) 7. **J07/Q10**

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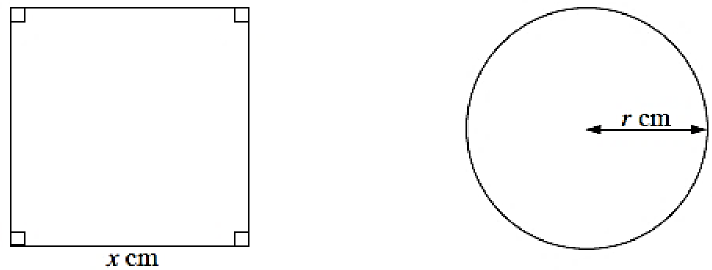
13 The equation of a curve is $y = (2x - 3)^3 - 6x$.

(i) Express $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of x. [3]

(ii) Find the x-coordinates of the two stationary points and determine the nature of each stationary point. [5]

Answers: (i) $6(2x-3)^2-6$, $24(2x-3)$; (ii) Minimum at $x=2$, Maximum at $x=1$. **N07/Q8**

14



A wire, 80 cm long, is cut into two pieces. One piece is bent to form a square of side x cm and the other piece is bent to form a circle of radius r cm (see diagram). The total area of the square and the circle is A cm^2.

(i) Show that $A = \dfrac{(\pi+4)x^2 - 160x + 1600}{\pi}$. [4]

(ii) Given that x and r can vary, find the value of x for which A has a stationary value. [4]

Answer: (ii) 11.2. **N08/Q7**

---

15 A solid rectangular block has a square base of side x cm. The height of the block is h cm and the total surface area of the block is 96 cm^2.

(i) Express h in terms of x and show that the volume, V cm^3, of the block is given by

$$V = 24x - \tfrac{1}{2}x^3.$$ [3]

Given that x can vary,

(ii) find the stationary value of V, [3]

(iii) determine whether this stationary value is a maximum or a minimum. [2]

Answers: (ii) 64; (iii) Maximum. **J10/12/Q8**

---

16 The equation of a curve is $y = \frac{1}{6}(2x-3)^3 - 4x$.

(i) Find $\dfrac{dy}{dx}$. [3]

(ii) Find the equation of the tangent to the curve at the point where the curve intersects the y-axis. [3]

(iii) Find the set of values of x for which $\frac{1}{6}(2x-3)^3 - 4x$ is an increasing function of x. [3]

J10/12/Q10

Answers: (i) $(2x-3)^2-4$; (ii) $2y = 10x - 9$; (iii) $x > 2\frac{1}{2}$, $x < \frac{1}{2}$.

17 The length, x metres, of a Green Anaconda snake which is t years old is given approximately by the formula

$$x = 0.7\sqrt{(2t-1)},$$

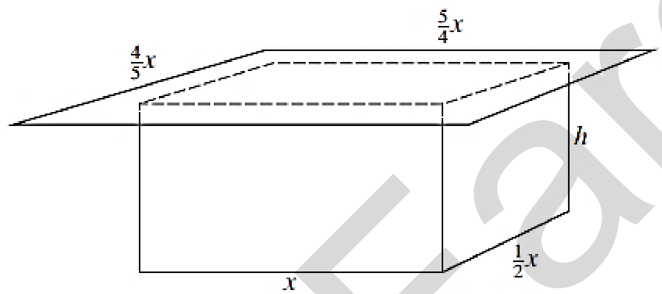
where $1 \leqslant t \leqslant 10$. Using this formula, find

(i) $\dfrac{dx}{dt}$, [2]

(ii) the rate of growth of a Green Anaconda snake which is 5 years old. [2]

Answers: **(i)** $\dfrac{0.7}{\sqrt{2t-1}}$; **(ii)** 0.233 metres/year. **N10/12/Q3**

18



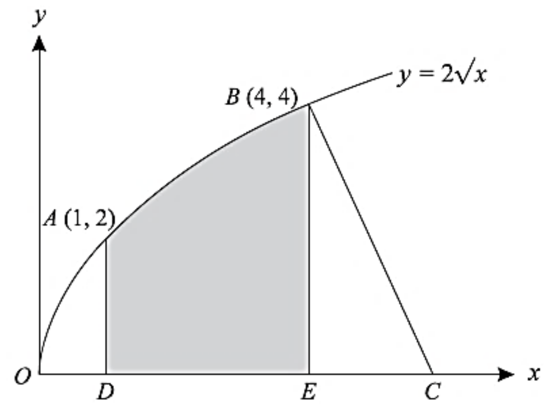
The diagram shows an open rectangular tank of height h metres covered with a lid. The base of the tank has sides of length x metres and $\frac{1}{2}x$ metres and the lid is a rectangle with sides of length $\frac{5}{4}x$ metres and $\frac{4}{5}x$ metres. When full the tank holds $4\,\text{m}^3$ of water. The material from which the tank is made is of negligible thickness. The external surface area of the tank together with the area of the top of the lid is $A\,\text{m}^2$.

(i) Express h in terms of x and hence show that $A = \frac{3}{2}x^2 + \dfrac{24}{x}$. [5]

(ii) Given that x can vary, find the value of x for which A is a minimum, showing clearly that A is a minimum and not a maximum. [5]

Answers: **(i)** $h = \dfrac{8}{x^2}$; **(ii)** 2. **N10/12/Q10**

19

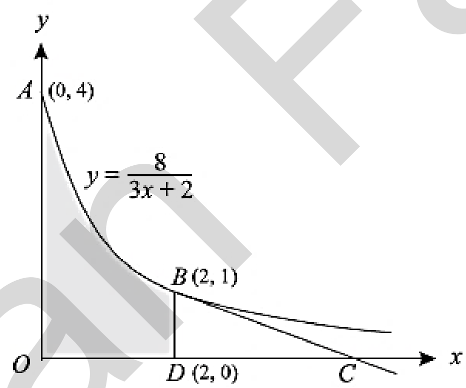


The diagram shows the points A (1, 2) and B (4, 4) on the curve $y = 2\sqrt{x}$. The line BC is the normal to the curve at B, and C lies on the x-axis. Lines AD and BE are perpendicular to the x-axis.

(i) Find the equation of the normal BC. [4]

Answers: (i) $y + 2x = 12$; **N02/Q10i**

20



The diagram shows points A (0, 4) and B (2, 1) on the curve $y = \dfrac{8}{3x+2}$. The tangent to the curve at B crosses the x-axis at C. The point D has coordinates (2, 0).

(i) Find the equation of the tangent to the curve at B and hence show that the area of triangle BDC is $\frac{4}{3}$. [6]

Answers: (i) $8y + 3x = 14$. **N03/Q9**

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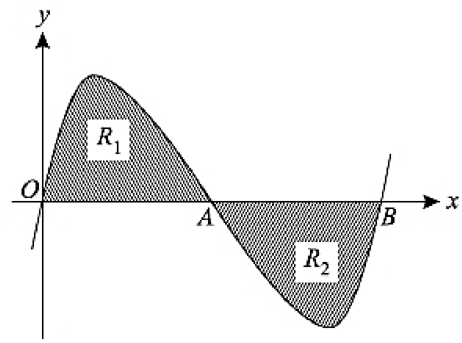
21 A curve is such that $\dfrac{dy}{dx} = \dfrac{6}{\sqrt{(4x-3)}}$ and P (3, 3) is a point on the curve.

(i) Find the equation of the normal to the curve at P, giving your answer in the form $ax + by = c$. [3]

Answers: (i) $x + 2y = 9$; (ii) $y = 3\sqrt{4x-3} - 6$. **N04/Q7**

23



The diagram shows the curve $y = x(x-1)(x-2)$, which crosses the x-axis at the points $O\,(0, 0)$, $A\,(1, 0)$ and $B\,(2, 0)$.

(i) The tangents to the curve at the points A and B meet at the point C. Find the x-coordinate of C. [5]

Answer: (i) $1\frac{2}{3}$. **N06/Q7**

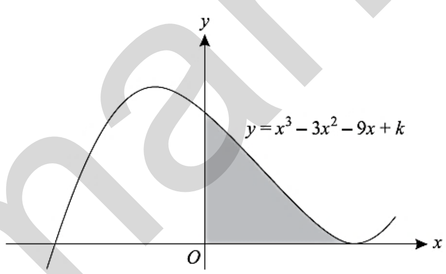
24

A curve is such that $\frac{dy}{dx} = \frac{4}{\sqrt{(6-2x)}}$, and $P\,(1, 8)$ is a point on the curve.

(i) The normal to the curve at the point P meets the coordinate axes at Q and at R. Find the coordinates of the mid-point of QR. [5]

Answers: (i) (8.5, 4.25); (ii) $y = 16 - 4\sqrt{6-2x}$. **J06/Q9**

25



The diagram shows the curve $y = x^3 - 3x^2 - 9x + k$, where k is a constant. The curve has a minimum point on the x-axis.

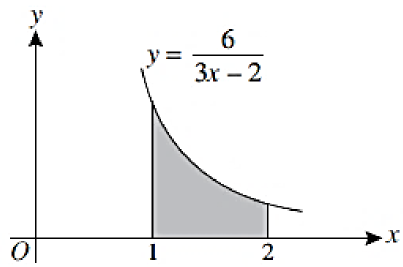
(i) Find the value of k. [4]

(ii) Find the coordinates of the maximum point of the curve. [1]

(iii) State the set of values of x for which $x^3 - 3x^2 - 9x + k$ is a decreasing function of x. [1]

Answers: (i) 27; (ii) (–1, 32); (iii) $-1 < x < 3$; (iv) 33.75. **J06/Q10**

27



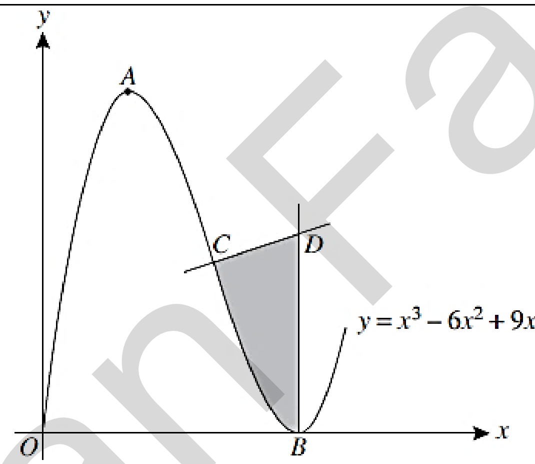
The diagram shows part of the curve $y = \dfrac{6}{3x-2}$.

(i) Find the gradient of the curve at the point where $x = 2$. [3]

J09/Q9

Answers: (i) $-\dfrac{9}{8}$; (ii) 9π

28



The diagram shows the curve $y = x^3 - 6x^2 + 9x$ for $x \geqslant 0$. The curve has a maximum point at A and a minimum point on the x-axis at B. The normal to the curve at $C\,(2, 2)$ meets the normal to the curve at B at the point D.

(i) Find the coordinates of A and B. [3]

(ii) Find the equation of the normal to the curve at C. [3]

J09/Q11

Answers: (i) $A\,(1, 4)$, $B\,(3, 0)$; (ii) $3y = x + 4$; (iii) $\dfrac{17}{12}$

29 The equation of a curve is $y = \sqrt{(5x+4)}$.

(i) Calculate the gradient of the curve at the point where $x = 1$. [3]

(ii) A point with coordinates (x, y) moves along the curve in such a way that the rate of increase of x has the constant value 0.03 units per second. Find the rate of increase of y at the instant when $x = 1$. [2]

(iii) Find the area enclosed by the curve, the x-axis, the y-axis and the line $x = 1$. [5]

Answers: (i) $\frac{5}{6}$; (ii) 0.025; (iii) 2.53 or $\frac{38}{15}$.

J03/Q10

30

The equation of a curve is $y = x^2 - 3x + 4$.

(i) Show that the whole of the curve lies above the x-axis. [3]

(ii) Find the set of values of x for which $x^2 - 3x + 4$ is a decreasing function of x. [1]

The equation of a line is $y + 2x = k$, where k is a constant.

(iii) In the case where $k = 6$, find the coordinates of the points of intersection of the line and the curve. [3]

(iv) Find the value of k for which the line is a tangent to the curve. [3]

Answers: (ii) $x < 1.5$; (iii) $(-1, 8)$ and $(2, 3)$; (iv) $3\frac{3}{4}$.

J05/Q10

31

A curve has equation $y = \frac{4}{3x-4}$ and $P\,(2, 2)$ is a point on the curve.

(i) Find the equation of the tangent to the curve at P. [4]

(ii) Find the angle that this tangent makes with the x-axis. [2]

Answers: **(i)** $y + 3x = 8$; **(ii)** 108.4° or 71.6°.

J11/12/Q4

32 The equation of a curve is $y = 4\sqrt{x} + \frac{2}{\sqrt{x}}$.

(i) Obtain an expression for $\frac{dy}{dx}$. [3]

(ii) A point is moving along the curve in such a way that the x-coordinate is increasing at a constant rate of 0.12 units per second. Find the rate of change of the y-coordinate when $x = 4$. [2]

N11/12/Q2

33 The volume of a solid circular cylinder of radius r cm is 250π cm^3.

(i) Show that the total surface area, S cm^2, of the cylinder is given by

$$S = 2\pi r^2 + \frac{500\pi}{r}.$$ [2]

(ii) Given that r can vary, find the stationary value of S. [4]

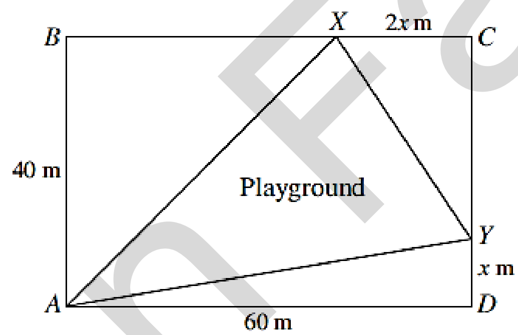
(iii) Determine the nature of this stationary value. [2]

Answers: **(ii)** $S = 150\pi$. **(iii)** Minimum. **J13/12/Q8**

34 Variables u, x and y are such that $u = 2x(y - x)$ and $x + 3y = 12$. Express u in terms of x and hence find the stationary value of u. [5]

Answer: 6 **J15/12/Q4**

35



The diagram shows a plan for a rectangular park $ABCD$, in which $AB = 40$ m and $AD = 60$ m. Points X and Y lie on BC and CD respectively and AX, XY and YA are paths that surround a triangular playground. The length of DY is x m and the length of XC is $2x$ m.

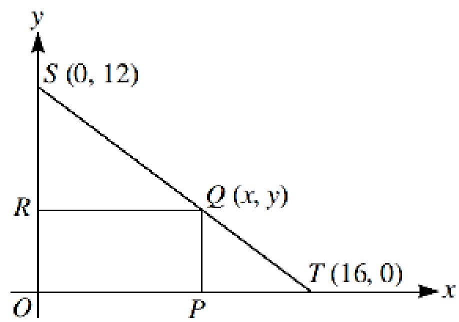
(i) Show that the area, A m^2, of the playground is given by

$$A = x^2 - 30x + 1200.$$ [2]

(ii) Given that x can vary, find the minimum area of the playground. [3]

Answers: **(i)** Proof; **(ii)** 975 m². **N12/12/Q3**

36



In the diagram, S is the point (0, 12) and T is the point (16, 0). The point Q lies on ST, between S and T, and has coordinates (x, y). The points P and R lie on the x-axis and y-axis respectively and $OPQR$ is a rectangle.

(i) Show that the area, A, of the rectangle $OPQR$ is given by $A = 12x - \frac{3}{4}x^2$. [3]

(ii) Given that x can vary, find the stationary value of A and determine its nature. [4]

Answers: **(ii)** A = 48 unit², maximum. **N13/12/Q6**

37 A curve has equation $y = \dfrac{12}{3 - 2x}$.

(i) Find $\dfrac{dy}{dx}$. [2]

A point moves along this curve. As the point passes through A, the x-coordinate is increasing at a rate of 0.15 units per second and the y-coordinate is increasing at a rate of 0.4 units per second.

(ii) Find the possible x-coordinates of A. [4]

Answer: 0 or 3 **N14/12/Q4**

38 The equation of a curve is $y = x^3 + ax^2 + bx$, where a and b are constants.

(i) In the case where the curve has no stationary point, show that $a^2 < 3b$. [3]

(ii) In the case where $a = -6$ and $b = 9$, find the set of values of x for which y is a decreasing function of x. [3]

Answer: **(ii)** $1 < x < 3$ **N14/12/Q6**

39

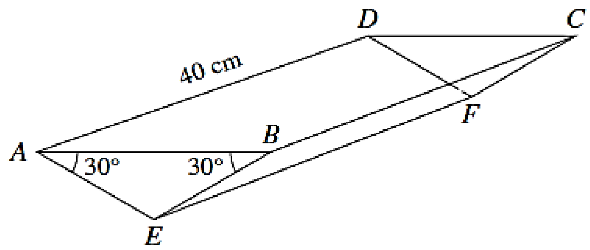


Fig. 1

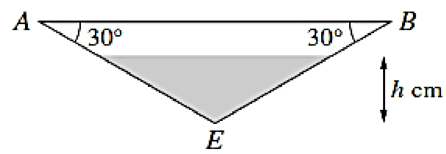


Fig. 2

Fig. 1 shows an open tank in the shape of a triangular prism. The vertical ends ABE and DCF are identical isosceles triangles. Angle ABE = angle $BAE = 30°$. The length of AD is 40 cm. The tank is fixed in position with the open top $ABCD$ horizontal. Water is poured into the tank at a constant rate of $200\ \text{cm}^3\ \text{s}^{-1}$. The depth of water, t seconds after filling starts, is h cm (see Fig. 2).

(i) Show that, when the depth of water in the tank is h cm, the volume, $V\ \text{cm}^3$, of water in the tank is given by $V = (40\sqrt{3})h^2$. [3]

(ii) Find the rate at which h is increasing when $h = 5$. [3]

Answer: **(ii)** 0.289

N15/12/Q3

40 A curve passes through the point $A\ (4, 6)$ and is such that $\frac{dy}{dx} = 1 + 2x^{-\frac{1}{2}}$. A point P is moving along the curve in such a way that the x-coordinate of P is increasing at a constant rate of 3 units per minute.

(i) Find the rate at which the y-coordinate of P is increasing when P is at A. [3]

(ii) Find the equation of the curve. [3]

(iii) The tangent to the curve at A crosses the x-axis at B and the normal to the curve at A crosses the x-axis at C. Find the area of triangle ABC. [5]

N15/13/Q9

Homework: Differentiation – Variants 11 & 13

1 The function f is defined by $f(x) = \frac{1}{x+1} + \frac{1}{(x+1)^2}$ for $x > -1$.

(i) Find $f'(x)$. [3]

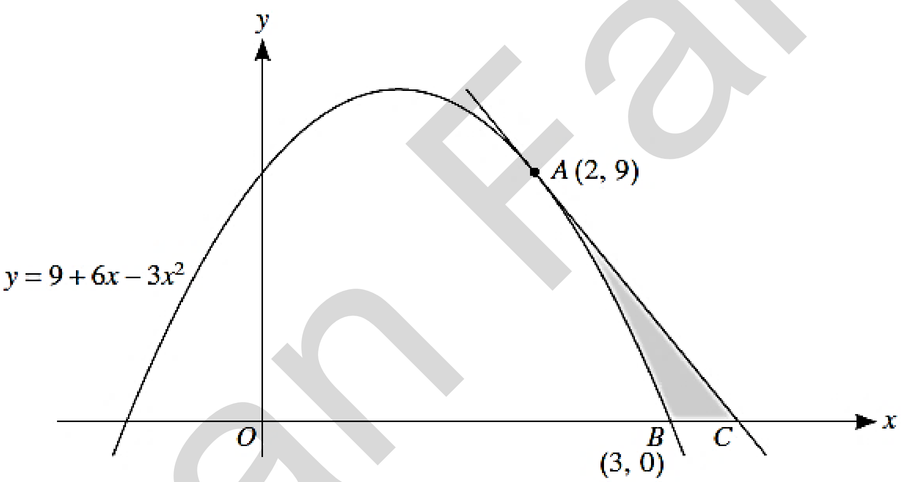
(ii) State, with a reason, whether f is an increasing function, a decreasing function or neither. [1]

The function g is defined by $g(x) = \frac{1}{x+1} + \frac{1}{(x+1)^2}$ for $x < -1$.

(iii) Find the coordinates of the stationary point on the curve $y = g(x)$. [4]

Answers: (i) $(x+1)^{-2} - 2(x+1)^{-3}$; (ii) $f'(x) < 0$ hence decreasing function; (iii) $(-3, -\frac{1}{4})$. 13/J15/8

2

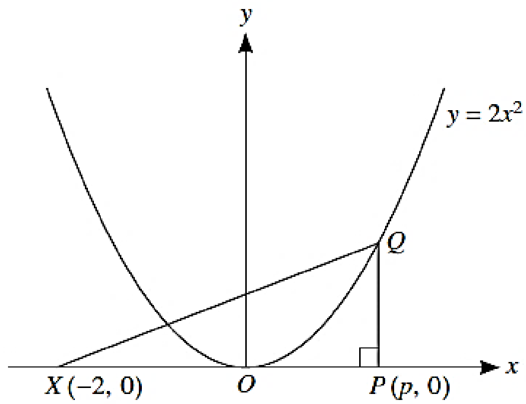


Points $A\,(2, 9)$ and $B\,(3, 0)$ lie on the curve $y = 9 + 6x - 3x^2$, as shown in the diagram. The tangent at A intersects the x-axis at C. Showing all necessary working,

(i) find the equation of the tangent AC and hence find the x-coordinate of C, [4]

Answers: (i) $y = -6x + 21$, $x = 3\frac{1}{2}$ 13/J15/10

3



The diagram shows the curve $y = 2x^2$ and the points $X(-2, 0)$ and $P(p, 0)$. The point Q lies on the curve and PQ is parallel to the y-axis.

(i) Express the area, A, of triangle XPQ in terms of p. [2]

The point P moves along the x-axis at a constant rate of 0.02 units per second and Q moves along the curve so that PQ remains parallel to the y-axis.

(ii) Find the rate at which A is increasing when $p = 2$. [3]

Answer: (i) $A = 2p^2 + p^3$ (ii) 0.4 — 11/J15/2

4 A piece of wire of length 24 cm is bent to form the perimeter of a sector of a circle of radius r cm.

(i) Show that the area of the sector, A cm^2, is given by $A = 12r - r^2$. [3]

(ii) Express A in the form $a - (r - b)^2$, where a and b are constants. [2]

(iii) Given that r can vary, state the greatest value of A and find the corresponding angle of the sector. [2]

Answer: (ii) $36 - (r - 6)^2$ (iii) $r = 6$, $\theta = 2$ — 11/J15/5

5 The equation of a curve is $y = x^3 + px^2$, where p is a positive constant.

(i) Show that the origin is a stationary point on the curve and find the coordinates of the other stationary point in terms of p. [4]

(ii) Find the nature of each of the stationary points. [3]

Another curve has equation $y = x^3 + px^2 + px$.

(iii) Find the set of values of p for which this curve has no stationary points. [3]

Answers: (i) and (ii) (0,0) minimum, $\left(\frac{-2p}{2}, \frac{4p^3}{27}\right)$ maximum (iii) $0 < p < 3$ — 11/J15/9

6 (i) Express $9x^2 - 12x + 5$ in the form $(ax+b)^2 + c$. [3]

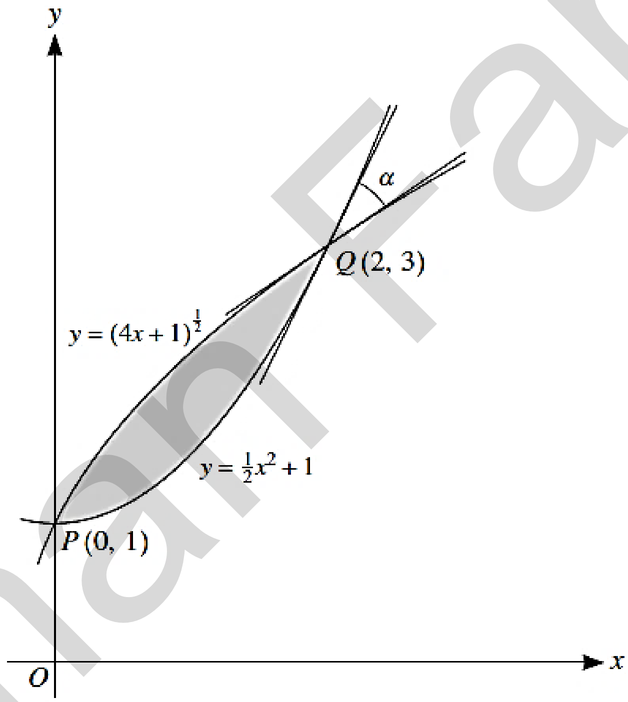
(ii) Determine whether $3x^3 - 6x^2 + 5x - 12$ is an increasing function, a decreasing function or neither. [3]

Answers: **(i)** $(3x-2)^2+1$; **(ii)** Increasing since derivative = $(3x-2)^2+1$ which is greater than 0 **13/N14/3**

7 (b) A point P travels along the curve $y = (7x^2+1)^{\frac{1}{3}}$ in such a way that the x-coordinate of P at time t minutes is increasing at a constant rate of 8 units per minute. Find the rate of increase of the y-coordinate of P at the instant when P is at the point (3, 4). [5]

(b) 7 **13/N14/10**

8



The diagram shows parts of the curves $y = (4x+1)^{\frac{1}{2}}$ and $y = \frac{1}{2}x^2 + 1$ intersecting at points $P(0, 1)$ and $Q(2, 3)$. The angle between the tangents to the two curves at Q is α.

(i) Find α, giving your answer in degrees correct to 3 significant figures. [6]

Answers: **(i)** 29.7° **11/N14/11**

9 The base of a cuboid has sides of length x cm and $3x$ cm. The volume of the cuboid is 288 cm^3.

(i) Show that the total surface area of the cuboid, $A\text{ cm}^2$, is given by

$$A = 6x^2 + \frac{768}{x}.$$ [3]

(ii) Given that x can vary, find the stationary value of A and determine its nature. [5]

Answers: (ii) 288, minimum. 13/J14/9

---

10 A curve has equation $y = \frac{4}{(3x+1)^2}$. Find the equation of the tangent to the curve at the point where the line $x = -1$ intersects the curve. [5]

Answer: $y = 3x + 4$. 11/J14/4

---

11 A curve is such that $\frac{dy}{dx} = x^{\frac{1}{2}} - x^{-\frac{1}{2}}$. The curve passes through the point $(4, \frac{2}{3})$.

(ii) Find $\frac{d^2y}{dx^2}$. [2]

(iii) Find the coordinates of the stationary point and determine its nature. [5]

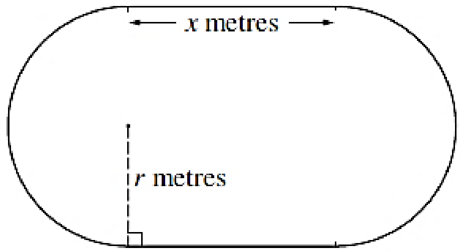
(ii) $\frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{2}x^{-\frac{3}{2}}$; (iii) (1, –2) minimum. 11/J14/12

---

12 A curve has equation $y = \frac{k^2}{x+2} + x$, where k is a positive constant. Find, in terms of k, the values of x for which the curve has stationary points and determine the nature of each stationary point. [8]

Answer: $-2 + k$: min, $-2 - k$: max 13/N13/9

13



The inside lane of a school running track consists of two straight sections each of length x metres, and two semicircular sections each of radius r metres, as shown in the diagram. The straight sections are perpendicular to the diameters of the semicircular sections. The perimeter of the inside lane is 400 metres.

(i) Show that the area, A m^2, of the region enclosed by the inside lane is given by $A = 400r - \pi r^2$. [4]

(ii) Given that x and r can vary, show that, when A has a stationary value, there are no straight sections in the track. Determine whether the stationary value is a maximum or a minimum. [5]

Answer: i) $A = 400r - \pi r^2$ iii) Max — 11/N13/8

14 The non-zero variables x, y and u are such that $u = x^2y$. Given that $y + 3x = 9$, find the stationary value of u and determine whether this is a maximum or a minimum value. [7]

Answers: 12; maximum. — 13/J13/6

15 A curve has equation $y = \mathrm{f}(x)$ and is such that $\mathrm{f}'(x) = 3x^{\frac{1}{2}} + 3x^{-\frac{1}{2}} - 10$.

(i) By using the substitution $u = x^{\frac{1}{2}}$, or otherwise, find the values of x for which the curve $y = \mathrm{f}(x)$ has stationary points. [4]

(ii) Find $\mathrm{f}''(x)$ and hence, or otherwise, determine the nature of each stationary point. [3]

(iii) It is given that the curve $y = \mathrm{f}(x)$ passes through the point $(4, -7)$. Find $\mathrm{f}(x)$. [4]

Answers: **(i)** $\frac{1}{9}$, 9; **(ii)** $\mathrm{f}''(x) = \frac{3}{2}x^{-\frac{1}{2}} - \frac{3}{2}x^{-\frac{3}{2}}$, Maximum at $x = \frac{1}{9}$, Minimum at $x = 9$; **(iii)** $\mathrm{f}(x) = 2x^{\frac{3}{2}} + 6x^{\frac{1}{2}} - 10x + 5$. — 11/J13/9

16 It is given that $\mathrm{f}(x) = \frac{1}{x^3} - x^3$, for $x > 0$. Show that f is a decreasing function. [3]

Answer: — 13/N12/2

17 A curve is such that

$$\frac{dy}{dx} = 2(3x+4)^{\frac{3}{2}} - 6x - 8.$$

(i) Find $\frac{d^2y}{dx^2}$. [2]

(ii) Verify that the curve has a stationary point when $x = -1$ and determine its nature. [2]

Answers: **(i)** $9(3x+4)^{\frac{1}{2}} - 6$; **(ii)** Minimum; **13/N12/8**

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18 An oil pipeline under the sea is leaking oil and a circular patch of oil has formed on the surface of the sea. At midday the radius of the patch of oil is 50 m and is increasing at a rate of 3 metres per hour. Find the rate at which the area of the oil is increasing at midday. [4]

Answer: 300π . **11/N12/3**

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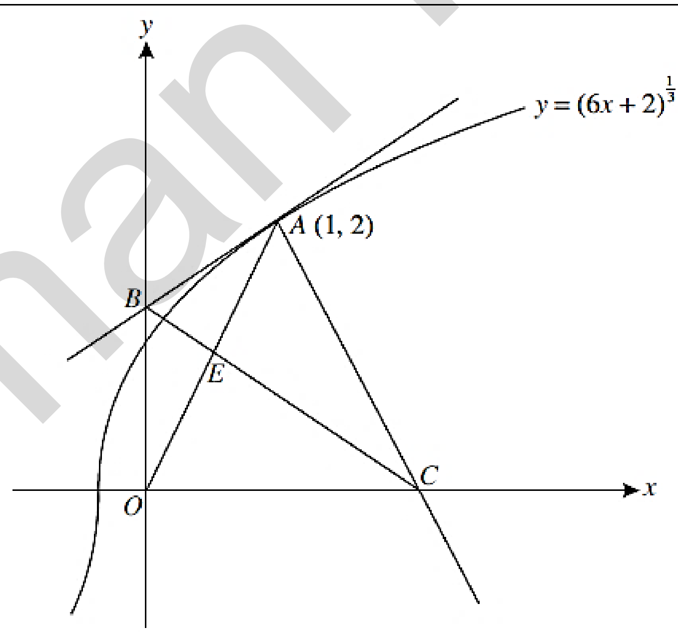
19 A curve has equation $y = 2x + \frac{1}{(x-1)^2}$. Verify that the curve has a stationary point at $x = 2$ and determine its nature. [5]

Answer: Minimum. **11/N12/5**

20



The diagram shows the curve $y = (6x+2)^{\frac{1}{3}}$ and the point $A\ (1, 2)$ which lies on the curve. The tangent to the curve at A cuts the y-axis at B and the normal to the curve at A cuts the x-axis at C.

(i) Find the equation of the tangent AB and the equation of the normal AC. [5]

(ii) Find the distance BC. [3]

(iii) Find the coordinates of the point of intersection, E, of OA and BC, and determine whether E is the mid-point of OA. [4]

Answers: **(i)** $y = x/2 + 3/2$, $y = -2x + 4$; **(ii)** $5/2$; **(iii)** $(6/11, 12/11)$, E is not the mid-point. **11/N12/11**

---

21 The curve $y = \frac{10}{2x+1} - 2$ intersects the x-axis at A. The tangent to the curve at A intersects the y-axis at C.

(i) Show that the equation of AC is $5y + 4x = 8$. [5]

(ii) Find the distance AC. [2]

Answers: **(i)** Proof **(ii)** $AC = 2.56$ **13/J12/7**

---

22 A watermelon is assumed to be spherical in shape while it is growing. Its mass, M kg, and radius, r cm, are related by the formula $M = kr^3$, where k is a constant. It is also assumed that the radius is increasing at a constant rate of 0.1 centimetres per day. On a particular day the radius is 10 cm and the mass is 3.2 kg. Find the value of k and the rate at which the mass is increasing on this day. [5]

Answers: $k = 0.0032$, 0.096. **11/J12/4**

---

23 It is given that a curve has equation $y = f(x)$, where $f(x) = x^3 - 2x^2 + x$.

(i) Find the set of values of x for which the gradient of the curve is less than 5. [4]

(ii) Find the values of $f(x)$ at the two stationary points on the curve and determine the nature of each stationary point. [5]

Answers: **(i)** $-\frac{2}{3} < x < 2$ **(ii)** $\frac{4}{27}$ (max), 0 (min) **11/J12/10**

---

24 A curve $y = f(x)$ has a stationary point at $P\,(3, -10)$. It is given that $f'(x) = 2x^2 + kx - 12$, where k is a constant.

(i) Show that $k = -2$ and hence find the x-coordinate of the other stationary point, Q. [4]

(ii) Find $f''(x)$ and determine the nature of each of the stationary points P and Q. [2]

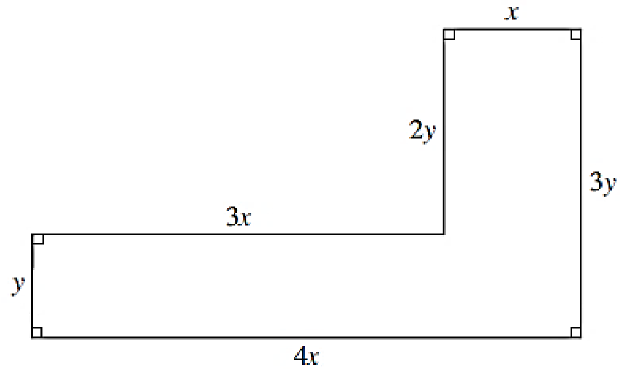
Answers: **(i)** $25 + p^2$; **(ii)** $25 + p^2 = 0$ has no real solutions; **(iii)** $p = \pm\sqrt{15}$. **13/N11/8**

---

25 A curve has equation $y = 3x^3 - 6x^2 + 4x + 2$. Show that the gradient of the curve is never negative. [3]

Answer: $(3x-2)^2 \geq 0$. **11/N11/2**

26



The diagram shows the dimensions in metres of an L-shaped garden. The perimeter of the garden is 48 m.

(i) Find an expression for y in terms of x. [1]

(ii) Given that the area of the garden is A m^2, show that $A = 48x - 8x^2$. [2]

(iii) Given that x can vary, find the maximum area of the garden, showing that this is a maximum value rather than a minimum value. [4]

Answers: **(i)** $y = \frac{1}{6}(48 - 8x)$; **(ii)** $A = 48x - 8x^2$; **(iii)** maximum area = 72m². **11/N11/7**

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27 **(a)** Differentiate $\frac{2x^3 + 5}{x}$ with respect to x. [3]

Answers: **(a)** $4x - 5x^{-2}$; **13/J11/4**

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28 The volume of a spherical balloon is increasing at a constant rate of 50 cm^3 per second. Find the rate of increase of the radius when the radius is 10 cm. [Volume of a sphere = $\frac{4}{3}\pi r^3$.] [4]

Answer: $\frac{1}{8\pi}$ or 0.0398. **11/J11/2**

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29 The variables x, y and z can take only positive values and are such that

$$z = 3x + 2y \quad \text{and} \quad xy = 600.$$

(i) Show that $z = 3x + \frac{1200}{x}$. [1]

(ii) Find the stationary value of z and determine its nature. [6]

Answers: **(ii)** 120, Minimum. **11/J11/6**

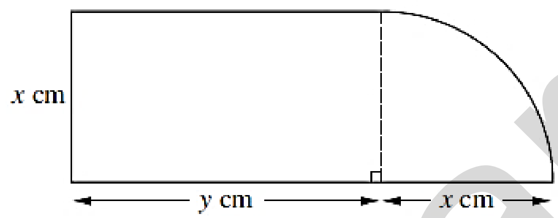
30 A curve has equation $y = \frac{1}{x-3} + x$.

(i) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. [2]

(ii) Find the coordinates of the maximum point A and the minimum point B on the curve. [5]

Answers: (i) $-\frac{1}{(x-3)^2}+1$, $\frac{2}{(x-3)^3}$; (ii) (2, 1), (4, 5). 13/N10/5

31



The diagram shows a metal plate consisting of a rectangle with sides x cm and y cm and a quarter-circle of radius x cm. The perimeter of the plate is 60 cm.

(i) Express y in terms of x. [2]

(ii) Show that the area of the plate, A cm^2, is given by $A = 30x - x^2$. [2]

Given that x can vary,

(iii) find the value of x at which A is stationary, [2]

(iv) find this stationary value of A, and determine whether it is a maximum or a minimum value. [2]

Answers: (i) $y = 30 - x - \frac{\pi x}{4}$; (iii) 15; (iv) 225, Maximum. 11/N10/8

32 The equation of a curve is $y = 3 + 4x - x^2$.

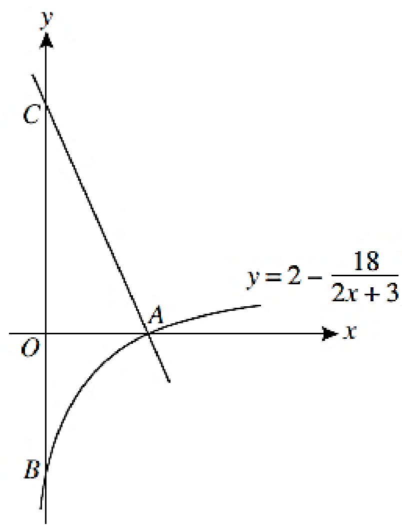
(i) Show that the equation of the normal to the curve at the point (3, 6) is $2y = x + 9$. [4]

(ii) Given that the normal meets the coordinate axes at points A and B, find the coordinates of the mid-point of AB. [2]

(iii) Find the coordinates of the point at which the normal meets the curve again. [4]

Answers: (ii) (–4.5, 2.25); (iii) (0.5, 4.75). 11/N10/10

33



The diagram shows part of the curve $y = 2 - \frac{18}{2x+3}$, which crosses the x-axis at A and the y-axis at B. The normal to the curve at A crosses the y-axis at C.

(i) Show that the equation of the line AC is $9x + 4y = 27$. [6]

(ii) Find the length of BC. [2]

11/J10/7

Answer: (ii) $10\frac{3}{4}$.

34 A curve has equation $y = \frac{8}{x} + 2x$.

(i) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. [3]

(ii) Find the coordinates of the stationary points and state, with a reason, the nature of each stationary point. [5]

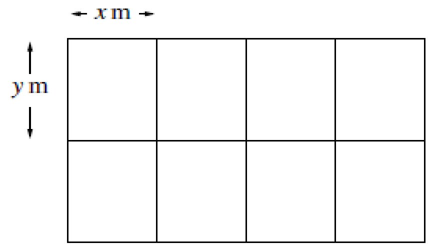
N15/11/Q5

35 **(i)** Express $3x^2 - 6x + 2$ in the form $a(x + b)^2 + c$, where a, b and c are constants. [3]

(ii) The function f, where $f(x) = x^3 - 3x^2 + 7x - 8$, is defined for $x \in \mathbb{R}$. Find $f'(x)$ and state, with a reason, whether f is an increasing function, a decreasing function or neither. [3]

N15/13/Q3

36



A farmer divides a rectangular piece of land into 8 equal-sized rectangular sheep pens as shown in the diagram. Each sheep pen measures x m by y m and is fully enclosed by metal fencing. The farmer uses 480 m of fencing.

(i) Show that the total area of land used for the sheep pens, $A\ \text{m}^2$, is given by

$$A = 384x - 9.6x^2.$$ [3]

(ii) Given that x and y can vary, find the dimensions of each sheep pen for which the value of A is a maximum. (There is no need to verify that the value of A is a maximum.) [3]

Answer: (ii) $x = 20,\ y = 24$ J16/11/Q5

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37 A curve has equation $y = 3x - \frac{4}{x}$ and passes through the points $A\,(1, -1)$ and $B\,(4, 11)$. At each of the points C and D on the curve, the tangent is parallel to AB. Find the equation of the perpendicular bisector of CD. [7]

Answer: $y = \frac{-1}{2}x$ J16/11/Q8

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38 A curve has equation $y = 8x + (2x - 1)^{-1}$. Find the values of x at which the curve has a stationary point and determine the nature of each stationary point, justifying your answers. [7]

Answer: x = ¼ maximum, x = ¾ minimum. J16/13/Q5

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39 The point $P\,(x, y)$ is moving along the curve $y = x^2 - \frac{10}{3}x^{\frac{3}{2}} + 5x$ in such a way that the rate of change of y is constant. Find the values of x at the points at which the rate of change of x is equal to half the rate of change of y. [7]

Answer: x = 9/4, 1. J16/13/Q7

40 The horizontal base of a solid prism is an equilateral triangle of side x cm. The sides of the prism are vertical. The height of the prism is h cm and the volume of the prism is 2000 cm^3.

(i) Express h in terms of x and show that the total surface area of the prism, $A\text{ cm}^2$, is given by

$$A = \frac{\sqrt{3}}{2}x^2 + \frac{24\,000}{\sqrt{3}}x^{-1}.$$ [3]

(ii) Given that x can vary, find the value of x for which A has a stationary value. [3]

(iii) Determine, showing all necessary working, the nature of this stationary value. [2]

Answer: **(ii)** 20 **(iii)** minimum **J17/11/Q6**

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41 A curve for which $\frac{dy}{dx} = 7 - x^2 - 6x$ passes through the point $(3, -10)$.

(i) Find the equation of the curve. [3]

13

(ii) Express $7 - x^2 - 6x$ in the form $a - (x+b)^2$, where a and b are constants. [2]

(iii) Find the set of values of x for which the gradient of the curve is positive. [3]

Answers: **(i)** $y = 7x - \frac{x^3}{3} - 3x^2 + 5$ **(ii)** $16 - (x+3)^2$ **(iii)** $-7 < x < 1$ or equivalent **J17/11/Q7**

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42 The line $3y + x = 25$ is a normal to the curve $y = x^2 - 5x + k$. Find the value of the constant k. [6]

Answer : $k = 11$. **J17/13/Q6**

---

43 A point is moving along the curve $y = 2x + \frac{5}{x}$ in such a way that the x-coordinate is increasing at a constant rate of 0.02 units per second. Find the rate of change of the y-coordinate when $x = 1$. [4]

Answer: -0.06 **J18/11/Q2**

---

44 (i) The tangent to the curve $y = x^3 - 9x^2 + 24x - 12$ at a point A is parallel to the line $y = 2 - 3x$. Find the equation of the tangent at A. [6]

(ii) The function f is defined by $f(x) = x^3 - 9x^2 + 24x - 12$ for $x > k$, where k is a constant. Find the smallest value of k for f to be an increasing function. [2]

Answers: **(i)** $y - 6 = -3(x - 3)$; **(ii)** Smallest value of k is 4. **J18/13/Q8**

45 The point $P(3, 5)$ lies on the curve $y = \frac{1}{x-1} - \frac{9}{x-5}$.

(i) Find the x-coordinate of the point where the normal to the curve at P intersects the x-axis. [5]

(ii) Find the x-coordinate of each of the stationary points on the curve and determine the nature of each stationary point, justifying your answers. [6]

Answers: **(i)** 13 **(ii)** –1, maximum; 2, minimum **N16/11/Q11**

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46 A curve has equation $y = 2x^{\frac{3}{2}} - 3x - 4x^{\frac{1}{2}} + 4$. Find the equation of the tangent to the curve at the point (4, 0). [4]

Answer: $y = 2(x - 4)$ **N17/11/Q1**

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47 A function f is defined by $f : x \mapsto x^3 - x^2 - 8x + 5$ for $x < a$. It is given that f is an increasing function. Find the largest possible value of the constant a. [4]

Answer: $a = -4/3$ **N17/11/Q2**

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48 Machines in a factory make cardboard cones of base radius r cm and vertical height h cm. The volume, V cm^3, of such a cone is given by $V = \frac{1}{3}\pi r^2 h$. The machines produce cones for which $h + r = 18$.

(i) Show that $V = 6\pi r^2 - \frac{1}{3}\pi r^3$. [1]

(ii) Given that r can vary, find the non-zero value of r for which V has a stationary value and show that the stationary value is a maximum. [4]

(iii) Find the maximum volume of a cone that can be made by these machines. [1]

Answers: **(ii)** $r = 12$ **(ii)** 288π **N17/11/Q4**

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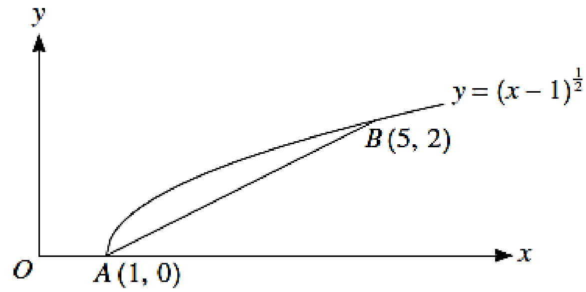
49 The function f is such that $f(x) = (2x - 1)^{\frac{3}{2}} - 6x$ for $\frac{1}{2} < x < k$, where k is a constant. Find the largest value of k for which f is a decreasing function. [5]

Answer: Largest value of k is $\frac{5}{2}$ **N17/13/Q4**

50



The diagram shows the curve $y = (x-1)^{\frac{1}{2}}$ and points A (1, 0) and B (5, 2) lying on the curve.

(i) Find the equation of the line AB, giving your answer in the form $y = mx + c$. [2]

(ii) Find, showing all necessary working, the equation of the tangent to the curve which is parallel to AB. [5]

(iii) Find the perpendicular distance between the line AB and the tangent parallel to AB. Give your answer correct to 2 decimal places. [3]

N17/13/Q11

Answers: **(i)** $y = \frac{1}{2}x - \frac{1}{2}$ **(ii)** $y = \frac{1}{2}x$ **(iii)** 0.45

---

51 A curve has equation $y = \frac{1}{2}(4x-3)^{-1}$. The point A on the curve has coordinates $\left(1, \frac{1}{2}\right)$.

(i) (a) Find and simplify the equation of the normal through A. [5]

(b) Find the x-coordinate of the point where this normal meets the curve again. [3]

(ii) A point is moving along the curve in such a way that as it passes through A its x-coordinate is decreasing at the rate of 0.3 units per second. Find the rate of change of its y-coordinate at A. [2]

N18/11/Q10

Answers: **(i)(a)** $y = \frac{x}{2}$ **(i)(b)** $\frac{-1}{4}$ **(ii)** 0.6

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52 The function f is defined by $f(x) = x^3 + 2x^2 - 4x + 7$ for $x \geqslant -2$. Determine, showing all necessary working, whether f is an increasing function, a decreasing function or neither. [4]

N18/11/Q2

Integration

Example 1

Find the value of c in each case where the equation of the curve and point on the curve are given.

a) $y = -5x + c$ $\quad(7, -2)$

b) $y = 6x + c$ $\quad(-4, -3)$

c) $y = 2x^2 + 3x + c$ $\quad(-1, -2)$

d) $y = -x^2 - 4x + c$ $\quad(5, 9)$

e) $y = 3x^3 - x^2 + 7x + c$ $\quad(-2, 8)$

$$\text{If } \frac{dy}{dx} = x^n \text{, then } y = \int x^n \, dx = \frac{x^{n+1}}{n+1} + c \; (n \neq -1)$$

$$\text{If } \frac{dy}{dx} = ax^n \text{, then } y = \int ax^n \, dx = \frac{ax^{n+1}}{n+1} + c \; (n \neq -1)$$

Example 2

Given $\frac{dy}{dx} = 9x^2 - 4x + 3x^{\frac{1}{2}} - 7$, find y.

Example 3

Given $f'(x) = x(3x - 4)^2$, find $f(x)$.

$$\int (ax + b)^n \, dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c, \text{ provided } n \neq -1$$

Example 4

Find $\int (4x - 3)^5 \, dx$.

Example 5

Find $\int \frac{4}{(1-2x)^7}\,dx$.

Example 6

$\int \frac{6}{\sqrt{(4x+9)}}\,dx$

Equation of the Curve

Example 7

Find the equation of the curve that passes through the point (4, –1) and where $\frac{dy}{dx} = \frac{5x^2+1}{\sqrt{x}}$.

Example 8

A curve is such that $f'(x) = \frac{10}{x^3} - 4$ and the point (–1, 2) lies on the curve. Find the equation of the curve.

Example 9

2020 Specimen Paper 1 Question 4

A curve has equation $y = f(x)$. It is given that $f'(x) = \frac{1}{\sqrt{x+6}} + \frac{6}{x^2}$ and that $f(3) = 1$.

Find $f(x)$. [5]

Example 10

November 2016/11 Question 10

A curve has equation $y = f(x)$ and it is given that $f'(x) = 3x^{\frac{1}{2}} - 2x^{-\frac{1}{2}}$. The point A is the only point on the curve at which the gradient is -1.

(i) Find the x-coordinate of A. [3]

(ii) Given that the curve also passes through the point (4, 10), find the y-coordinate of A, giving your answer as a fraction. [6]

Example 11

November 2016/12 Question 1

A curve is such that $\frac{dy}{dx} = \frac{8}{\sqrt{(4x+1)}}$. The point (2, 5) lies on the curve. Find the equation of the curve. [4]

Example 12

November 2016/13 Question 10

A curve is such that $\frac{dy}{dx} = \frac{2}{a}x^{-\frac{1}{2}} + ax^{-\frac{3}{2}}$, where a is a positive constant. The point $A\,(a^2, 3)$ lies on the curve. Find, in terms of a,

(i) the equation of the tangent to the curve at A, simplifying your answer, [3]

(ii) the equation of the curve. [4]

It is now given that $B\,(16, 8)$ also lies on the curve.

(iii) Find the value of a and, using this value, find the distance AB. [5]

Example 13

A curve is such that $\frac{d^2y}{dx^2} = -6x$ and the curve has a maximum point at (1, 2).
Find the equation of the curve.

Example 14

The equation of a curve is such that $\frac{dy}{dx} = \frac{3}{\sqrt{x}} - x$. Given that the curve passes through the point (4, 6), find the equation of the curve.

[Cambridge AS & A Level Mathematics 9709, Paper 12 Q1 November 2009]

Example 15

A curve is such that $\frac{dy}{dx} = 4 - x$ and the point P(2, 9) lies on the curve. The normal to the curve at P meets the curve again at Q. Find

(I) the equation of the curve,

(II) the equation of the normal to the curve at P,

(III) the co-ordinates of Q.

[Cambridge AS & A Level Mathematics 9709, Paper 1 Q9 November 2007]

Example 16

A curve is such that $\frac{dy}{dx} = \frac{4}{\sqrt{6-2x}}$, and P(1, 8) is a point on the curve.

(i) The normal to the curve at the point P meets the co-ordinate axes at Q and at R. Find the co-ordinates of the mid-point of QR.

(ii) Find the equation of the curve.

[Cambridge AS & A Level Mathematics 9709, Paper 1 Q9 June 2006]

Definite Integral

$$\int_a^b f'(x)\,dx = [f(x)]_a^b = f(b) - f(a)$$

Example 17

a) Find $\int (3x-2)^8\,dx$.

b) Hence find $\int_0^1 (3x-2)^8\,dx$.

Example 18

Find $\int_1^4 \frac{x^2 + x^3}{\sqrt{x}}\,dx$.

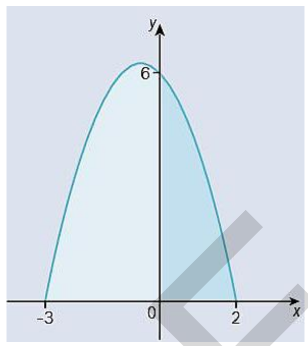
Finding the area using integration

The area between the curve $y = f(x)$ and the x-axis, bounded by the line $x = a$ and the line $x = b$ is given by

$$\text{Area} = \int_a^b f(x)\,dx = \int_a^b y\,dx$$

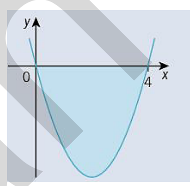
Example 19

Find the area bounded by the curve with equation $y = (2 - x)(3 + x)$, the positive x-axis and the y-axis.



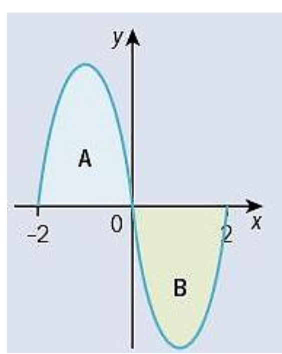
Example 20

Find the area bounded by the curve with equation $y = x(x - 4)$ and the x-axis.



Example 21

Find the area bounded by the curve with equation $y = x(x + 2)(x - 2)$ and the x-axis.



Example 22

A curve has equation $y = \frac{4}{\sqrt{x}}$.

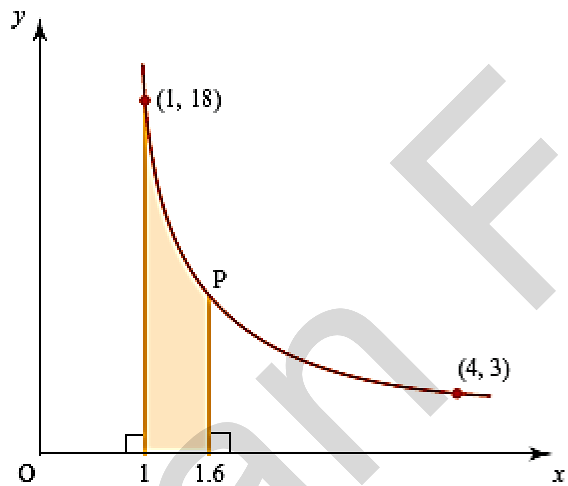
(i) The normal to the curve at the point (4, 2) meets the x axis at P and the y axis at Q. Find the length of PQ, correct to 3 significant figures.

(ii) Find the area of the region enclosed by the curve, the x axis and the lines $x = 1$ and $x = 4$.

[Cambridge AS & A Level Mathematics 9709, Paper 1 Q9 June 2005]

Example 24

The diagram shows a curve for which $\frac{dy}{dx} = -\frac{k}{x^3}$, where k is a constant. The curve passes through the points (1, 18) and (4, 3).



(i) Show, by integration, that the equation of the curve is $y = \frac{16}{x^2} + 2$.

The point P lies on the curve and has x co-ordinate 1.6.

(ii) Find the area of the shaded region.

[Cambridge AS & A Level Mathematics 9709, Paper 1 Q9 June 2008]

Example 25

A curve is such that $\frac{dy}{dx} = \frac{16}{x^3}$, and (1, 4) is a point on the curve.

(I) Find the equation of the curve.

(II) A line with gradient $-\frac{1}{2}$ is a normal to the curve. Find the equation of this normal, giving your answer in the form $ax + by = c$.

(III) Find the area of the region enclosed by the curve, the x axis and the lines $x = 1$ and $x = 2$.

[Cambridge AS & A Level Mathematics 9709, Paper 1 Q10 November 2005]

Example 26

The equation of a curve is $y = 2x + \frac{8}{x^2}$.

(I) Obtain expressions for $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

(II) Find the co-ordinates of the stationary point on the curve and determine the nature of the stationary point.

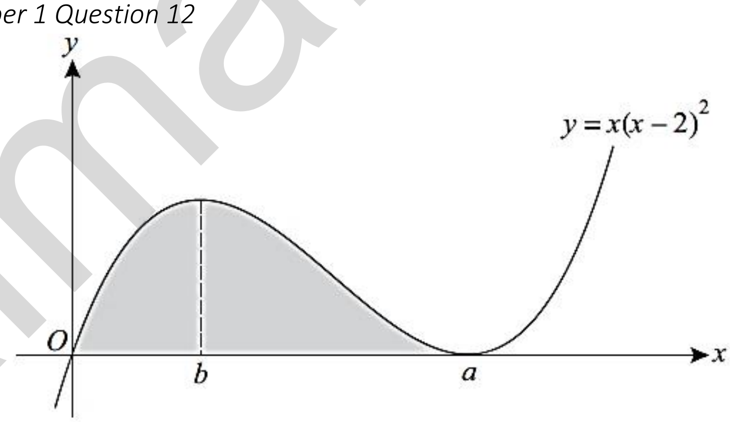
(III) Show that the normal to the curve at the point (–2, –2) intersects the x axis at the point (–10, 0).

(IV) Find the area of the region enclosed by the curve, the x axis and the lines $x = 1$ and $x = 2$.

[Cambridge AS & A Level Mathematics 9709, Paper 1 Q10 June 2007]

Example 27

2020 Specimen Paper 1 Question 12



The diagram shows the curve with equation $y = x(x-2)^2$. The minimum point on the curve has coordinates $(a, 0)$ and the x-coordinate of the maximum point is b, where a and b are constants.

(a) State the value of a. [1]

(b) Calculate the value of b. [4]

(c) Find the area of the shaded region. [4]

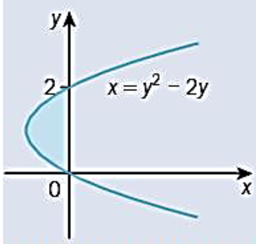
Area between curve and y axis

The area between the curve $x = f(y)$, the **y-axis**, and the lines $y = a$ and $y = b$ is given by:

$$\text{Area} = \int_{y=a}^{y=b} x \, dy \text{ or } \int_{y=a}^{y=b} f(y) \, dy$$

Example 28

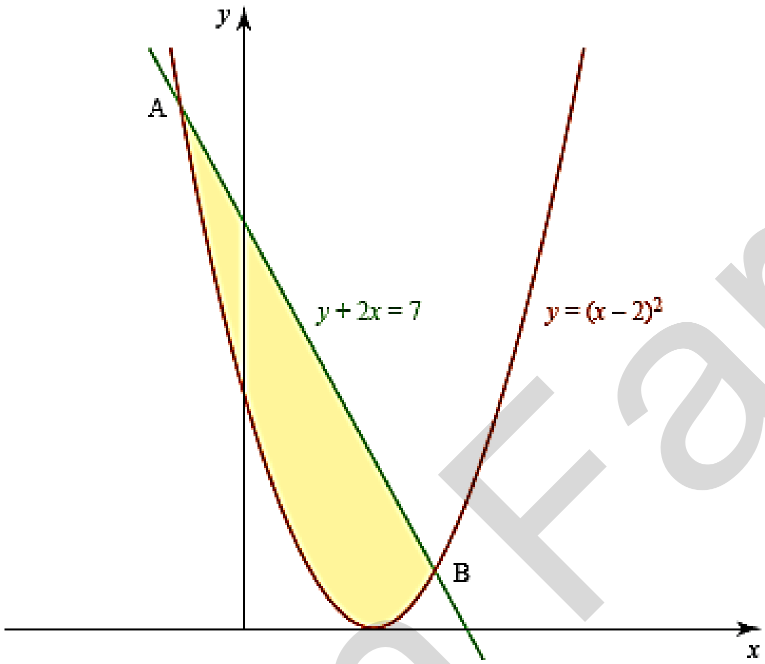
Find the area bounded by the curve with equation $x = y^2 - 2y$ and the y-axis as shown in the diagram.



Area between two curves

Example 29

The diagram shows the curve $y = (x-2)^2$ and the line $y + 2x = 7$, which intersect at points A and B.

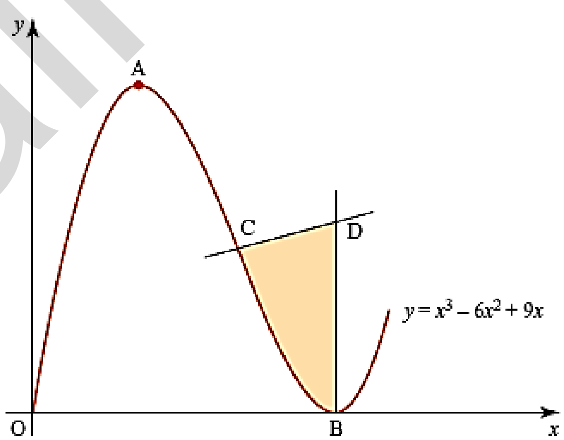


Find the area of the shaded region.

[Cambridge AS & A Level Mathematics 9709, Paper 12 Q9 June 2010]

Example 30

The diagram shows the curve $y = x^3 - 6x^2 + 9x$ for $x \geqslant 0$. The curve has a maximum point at A and a minimum point on the x axis at B. The normal to the curve at C(2, 2) meets the normal to the curve at B at the point D.



(I) Find the co-ordinates of A and B.

(II) Find the equation of the normal to the curve at C.

(III) Find the area of the shaded region.

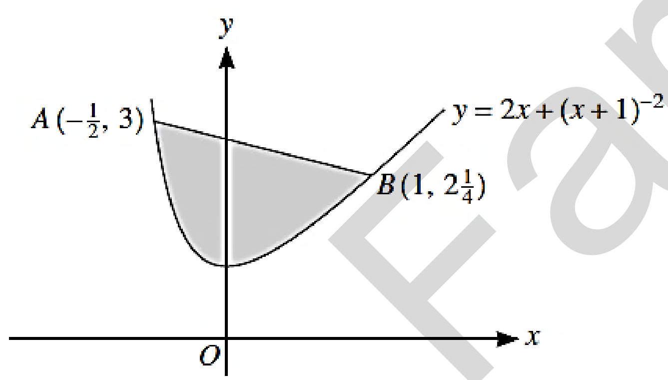
[Cambridge AS & A Level Mathematics 9709, Paper 1 Q11 June 2009]

Example 31

November 2015/13 Question 10

The function f is defined by $f(x) = 2x + (x+1)^{-2}$ for $x > -1$.

(i) Find $f'(x)$ and $f''(x)$ and hence verify that the function f has a minimum value at $x = 0$. [4]



The points $A(-\frac{1}{2}, 3)$ and $B(1, 2\frac{1}{4})$ lie on the curve $y = 2x + (x+1)^{-2}$, as shown in the diagram.

(ii) Find the distance AB. [2]

(iii) Find, showing all necessary working, the area of the shaded region. [6]

Improper Integrals

Improper integrals can be one of two types:

(1) Integrals where at least one of the limits is infinite,

e.g. **i)** $\displaystyle\int_1^{\infty} \frac{1}{x^2\sqrt{x}}\,dx$, e.g. **ii)** $\displaystyle\int_{-\infty}^{-2} \frac{2}{x^5}\,dx$.

(2) Integrals where the function to be integrated is not defined at a point in the interval of integration.

In this case, we will restrict our consideration to examples where the function is not defined at one end of the interval, e.g. $\displaystyle\int_{-2}^{0} \frac{2}{x^5}\,dx$ where $\frac{2}{x^5}$ is not defined at $x = 0$.

For **(1) i)**

We define the **improper integral** $\displaystyle\int_a^{\infty} f(x)\,dx$ as $\displaystyle\lim_{b\to\infty}\int_a^{b} f(x)\,dx$, provided the limit exists.

ii)

We define the **improper integral** $\displaystyle\int_{-\infty}^{b} f(x)\,dx$ as $\displaystyle\lim_{a\to-\infty}\int_a^{b} f(x)\,dx$, provided the limit exists.

For **(2)**

When $f(x)$ is defined for $0 < x < b$, but $f(x)$ is not defined when $x = 0$, then the **improper integral** $\displaystyle\int_0^{b} f(x)\,dx = \lim_{a\to 0^+}\int_a^{b} f(x)\,dx$, provided the limit exists.

Note: $\lim_{a\to 0^+}$ denotes 'a tends to 0 from just above 0'.

Example 32

Example 18

Show that the improper integral $\displaystyle\int_1^{\infty} \frac{1}{x^2\sqrt{x}}\,dx$ has a value and find that value.

Example 33

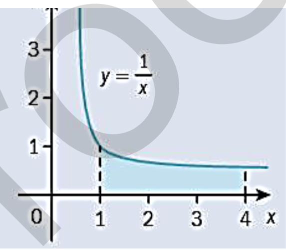
Show that only one of the following improper integrals has a finite value and find that value.

a) $\displaystyle\int_{-2}^{0} \frac{2}{x^5}\,dx$ **b)** $\displaystyle\int_{-\infty}^{-2} \frac{2}{x^5}\,dx$

The volume of a solid of revolution generated by rotating the curve $y = \mathrm{f}(x)$ between $x = a$ and $x = b$ through 360° about the x-axis is given by $V_x = \int_a^b \pi y^2 \,\mathrm{d}x$.

Example 34

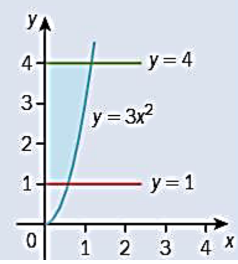
Find the volume obtained when the shaded region is rotated through 360° about the x-axis.



Similarly, the volume of a solid of revolution generated by rotating the curve $x = \mathrm{f}(y)$ between $y = a$ and $y = b$ through 360° about the y-axis is given by $V_y = \int_a^b \pi x^2 \,\mathrm{d}y$.

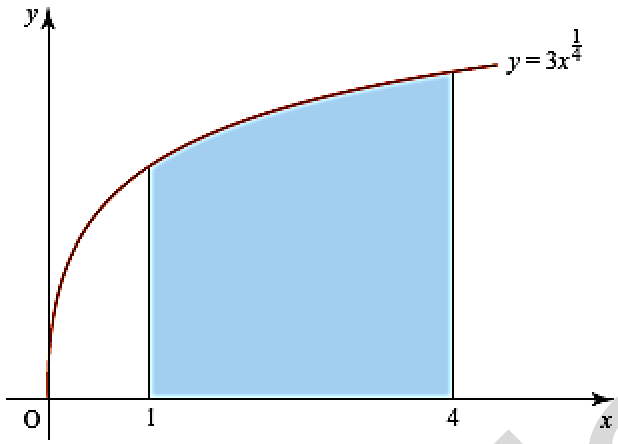
Example 35

The diagram shows the line $y = 1$, the line $y = 4$ and part of the curve $y = 3x^2$.
The shaded region is rotated through 360° about the y-axis.
Find the exact value of the volume of revolution obtained.



Example 36

The diagram shows the curve $y = 3x^{\frac{1}{4}}$. The shaded region is bounded by the curve, the x axis and the lines $x = 1$ and $x = 4$.

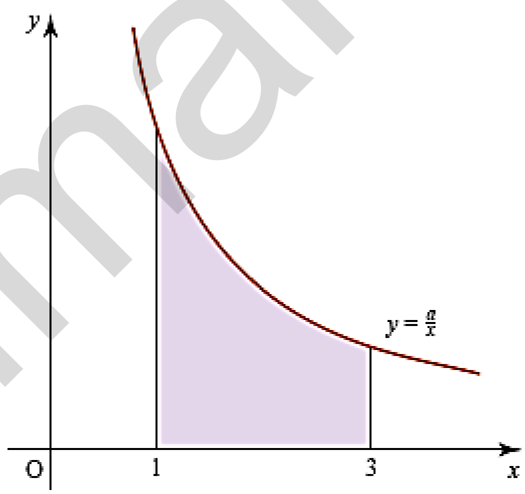


Find the volume of the solid obtained when this shaded region is rotated completely about the x axis, giving your answer in terms of π.

[Cambridge AS & A Level Mathematics 9709, Paper 1 Q2 June 2007]

Example 37

The diagram shows part of the curve $y = \frac{a}{x}$, where a is a positive constant.

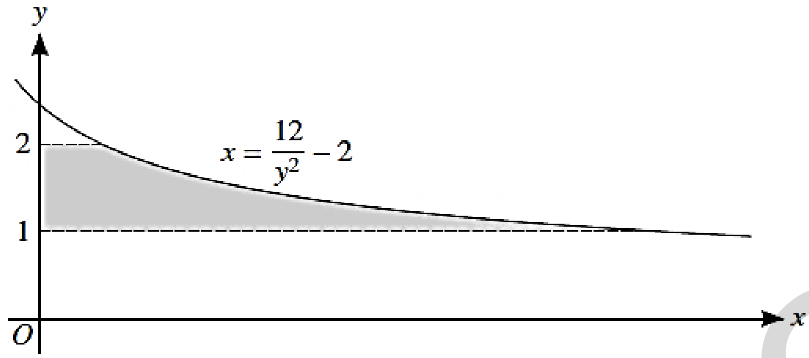


Given that the volume obtained when the shaded region is rotated through 360° about the x axis is 24π, find the value of a.

[Cambridge AS & A Level Mathematics 9709, Paper 12 Q2 June 2010]

Example 38

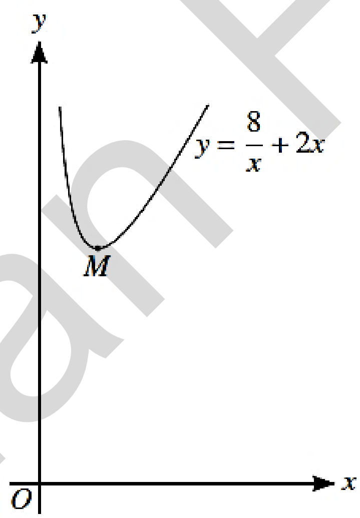
June 2016/11 Question 3



The diagram shows part of the curve $x = \frac{12}{y^2} - 2$. The shaded region is bounded by the curve, the y-axis and the lines $y = 1$ and $y = 2$. Showing all necessary working, find the volume, in terms of π, when this shaded region is rotated through 360° about the y-axis. [5]

Example 39

June 2016/12 Question 10



The diagram shows the part of the curve $y = \frac{8}{x} + 2x$ for $x > 0$, and the minimum point M.

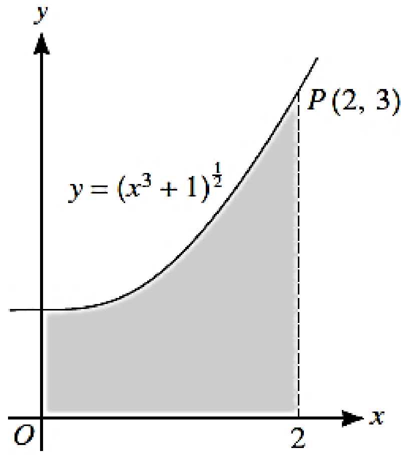
(i) Find expressions for $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ and $\int y^2\,dx$. [5]

(ii) Find the coordinates of M and determine the coordinates and nature of the stationary point on the part of the curve for which $x < 0$. [5]

(iii) Find the volume obtained when the region bounded by the curve, the x-axis and the lines $x = 1$ and $x = 2$ is rotated through 360° about the x-axis. [2]

Example 40

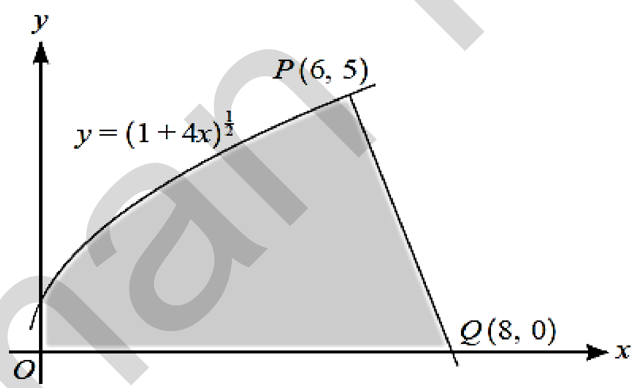
June 2016/13 Question 2



The diagram shows part of the curve $y = (x^3 + 1)^{\frac{1}{2}}$ and the point $P\,(2, 3)$ lying on the curve. Find, showing all necessary working, the volume obtained when the shaded region is rotated through 360° about the x-axis. [4]

Example 41

November 2015/11 Question 11



The diagram shows part of the curve $y = (1 + 4x)^{\frac{1}{2}}$ and a point $P\,(6, 5)$ lying on the curve. The line PQ intersects the x-axis at $Q\,(8, 0)$.

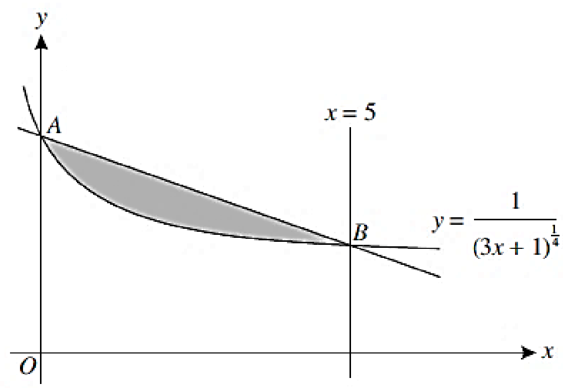
(i) Show that PQ is a normal to the curve. [5]

(ii) Find, showing all necessary working, the exact volume of revolution obtained when the shaded region is rotated through 360° about the x-axis. [7]

[In part **(ii)** you may find it useful to apply the fact that the volume, V, of a cone of base radius r and vertical height h, is given by $V = \frac{1}{3}\pi r^2 h$.]

Example 42

N10/12/Q11



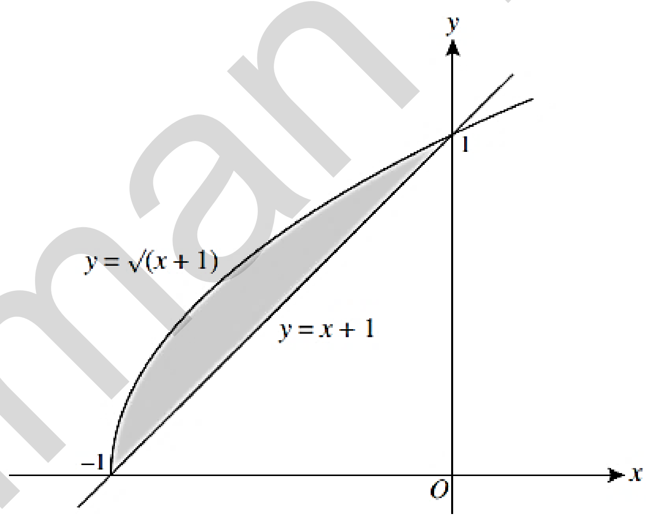
The diagram shows part of the curve $y = \dfrac{1}{(3x+1)^{\frac{1}{4}}}$. The curve cuts the y-axis at A and the line $x = 5$ at B.

(i) Show that the equation of the line AB is $y = -\frac{1}{10}x + 1$. [4]

(ii) Find the volume obtained when the shaded region is rotated through 360° about the x-axis. [9]

Example 43

– N11/13/Q10



The diagram shows the line $y = x + 1$ and the curve $y = \sqrt{(x + 1)}$, meeting at $(-1, 0)$ and $(0, 1)$.

(i) Find the area of the shaded region. [5]

(ii) Find the volume obtained when the shaded region is rotated through 360° about the y-axis. [7]

Homework: Integration Variant 12

1 The gradient at any point (x, y) on a curve is $\sqrt{(1+2x)}$. The curve passes through the point $(4, 11)$. Find

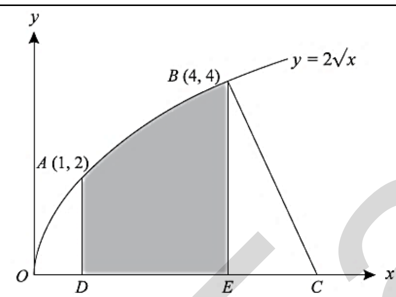
(i) the equation of the curve, [4]

(ii) the point at which the curve intersects the y-axis. [2]

N02/Q4

Answers: (i) $y = \frac{1}{3}(1+2x)^{\frac{3}{2}} + 2$; (ii) $\frac{7}{3}$.

2



The diagram shows the points $A\,(1, 2)$ and $B\,(4, 4)$ on the curve $y = 2\sqrt{x}$. The line BC is the normal to the curve at B, and C lies on the x-axis. Lines AD and BE are perpendicular to the x-axis.

(i) Find the equation of the normal BC. [4]

(ii) Find the area of the shaded region. [4]

Answers: (i) $y + 2x = 12$; (ii) $9\frac{1}{3}$.

N02/Q10

3 A curve is such that $\frac{dy}{dx} = 3x^2 - 4x + 1$. The curve passes through the point $(1, 5)$.

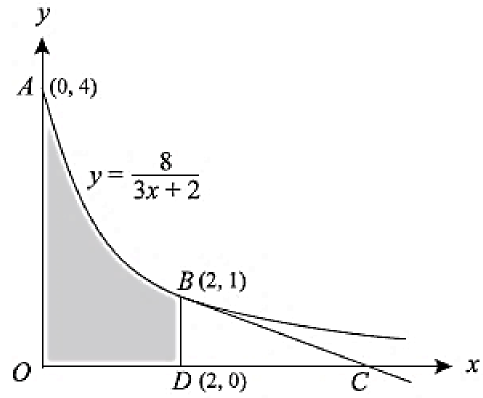
(i) Find the equation of the curve. [3]

(ii) Find the set of values of x for which the gradient of the curve is positive. [3]

N03/Q4

Answers: (i) $y = x^3 - 2x^2 + x + 5$; (ii) $x < \frac{1}{3}$ and $x > 1$.

4



The diagram shows points A (0, 4) and B (2, 1) on the curve $y = \frac{8}{3x+2}$. The tangent to the curve at B crosses the x-axis at C. The point D has coordinates (2, 0).

(i) Find the equation of the tangent to the curve at B and hence show that the area of triangle BDC is $\frac{4}{3}$. [6]

(ii) Show that the volume of the solid formed when the shaded region $ODBA$ is rotated completely about the x-axis is 8π. [5]

Answers: (i) $8y + 3x = 14$.

N03/Q9

5 A curve is such that $\frac{dy}{dx} = \frac{6}{\sqrt{(4x-3)}}$ and P (3, 3) is a point on the curve.

(i) Find the equation of the normal to the curve at P, giving your answer in the form $ax + by = c$. [3]

(ii) Find the equation of the curve. [4]

Answers: (i) $x + 2y = 9$; (ii) $y = 3\sqrt{4x-3} - 6$.

N04/Q7

6 A curve has equation $y = x^2 + \frac{2}{x}$.

(i) Write down expressions for $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. [3]

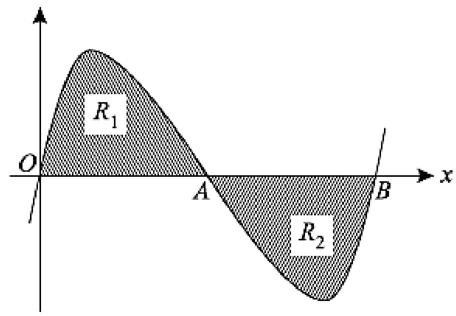
(ii) Find the coordinates of the stationary point on the curve and determine its nature. [4]

(iii) Find the volume of the solid formed when the region enclosed by the curve, the x-axis and the lines $x = 1$ and $x = 2$ is rotated completely about the x-axis. [6]

N04/Q10

Answers: (i) $2x - \frac{2}{x^2}$, $2 + \frac{4}{x^3}$ (ii) (1, 3), Minimum point; (iii) 14.2π or 44.6.

7



The diagram shows the curve $y = x(x - 1)(x - 2)$, which crosses the x-axis at the points $O\,(0, 0)$, $A\,(1, 0)$ and $B\,(2, 0)$.

(i) The tangents to the curve at the points A and B meet at the point C. Find the x-coordinate of C. [5]

(ii) Show by integration that the area of the shaded region R_1 is the same as the area of the shaded region R_2. [4]

N06/Q7

Answer: (i) $1\frac{2}{3}$.

---

8 Find the area of the region enclosed by the curve $y = 2\sqrt{x}$, the x-axis and the lines $x = 1$ and $x = 4$. [4]

N07/Q2

Answer: $9\frac{1}{3}$.

---

9 A curve is such that $\frac{dy}{dx} = 4 - x$ and the point $P\,(2, 9)$ lies on the curve. The normal to the curve at P meets the curve again at Q. Find

(i) the equation of the curve, [3]

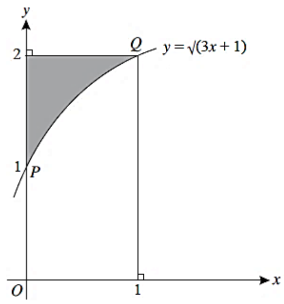
(ii) the equation of the normal to the curve at P, [3]

(iii) the coordinates of Q. [3]

N07/Q9

Answers: **(i)** $y = 4x - \frac{1}{2}x^2 + 3$; **(ii)** $2y + x = 20$; **(iii)** $(7, 6.5)$.

10



The diagram shows the curve $y = \sqrt{(3x + 1)}$ and the points $P\,(0,\ 1)$ and $Q\,(1,\ 2)$ on the curve. The shaded region is bounded by the curve, the y-axis and the line $y = 2$.

(i) Find the area of the shaded region. [4]

(ii) Find the volume obtained when the shaded region is rotated through 360° about the x-axis. [4]

Tangents are drawn to the curve at the points P and Q.

(iii) Find the acute angle, in degrees correct to 1 decimal place, between the two tangents. [4]

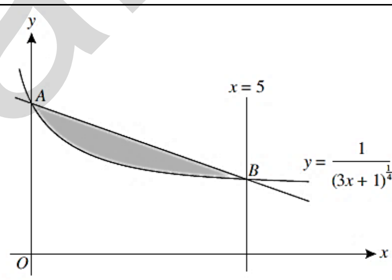
Answers: (i) $\frac{4}{9}$; (ii) 4.71; (iii) 19.4°. **N08/Q9**

---

11 The equation of a curve is such that $\frac{dy}{dx} = \frac{3}{\sqrt{x}} - x$. Given that the curve passes through the point (4, 6), find the equation of the curve. [4]

Answer: $y = 6\sqrt{x} - \frac{1}{2}x^2 + 2$. **N09/Q1**

12



The diagram shows part of the curve $y = \frac{1}{(3x+1)^{\frac{1}{4}}}$. The curve cuts the y-axis at A and the line $x = 5$ at B.

(i) Show that the equation of the line AB is $y = -\frac{1}{10}x + 1$. [4]

(ii) Find the volume obtained when the shaded region is rotated through 360° about the x-axis. [9]

Answer: (ii) $\frac{11}{12}\pi$. **N10/12/Q11**

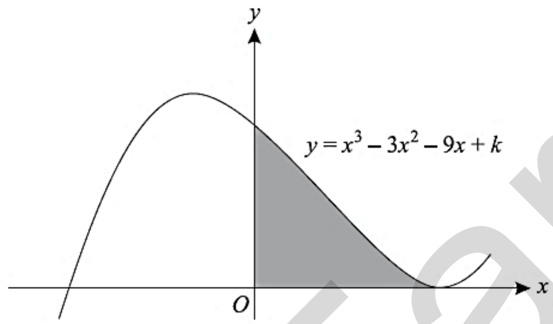
13 A curve is such that $\frac{dy}{dx} = \frac{4}{\sqrt{(6-2x)}}$, and $P(1, 8)$ is a point on the curve.

(i) The normal to the curve at the point P meets the coordinate axes at Q and at R. Find the coordinates of the mid-point of QR. [5]

(ii) Find the equation of the curve. [4]

Answers: **(i)** (8.5, 4.25); **(ii)** $y = 16 - 4\sqrt{6-2x}$. **J06/Q9**

14



The diagram shows the curve $y = x^3 - 3x^2 - 9x + k$, where k is a constant. The curve has a minimum point on the x-axis.

(i) Find the value of k. [4]

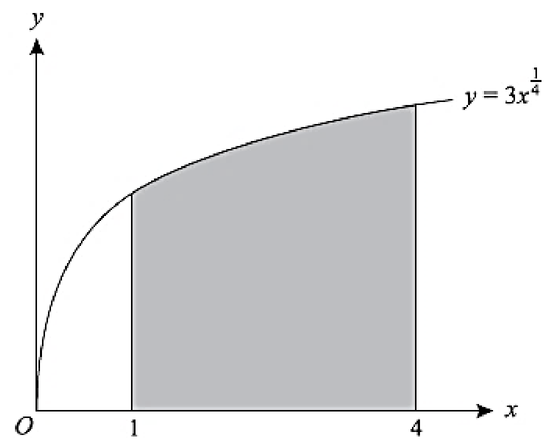
(ii) Find the coordinates of the maximum point of the curve. [1]

(iii) State the set of values of x for which $x^3 - 3x^2 - 9x + k$ is a decreasing function of x. [1]

(iv) Find the area of the shaded region. [4]

Answers: **(i)** 27; **(ii)** (−1, 32); **(iii)** $-1 < x < 3$; **(iv)** 33.75. **J06/Q10**

15



The diagram shows the curve $y = 3x^{\frac{1}{4}}$. The shaded region is bounded by the curve, the x-axis and the lines $x = 1$ and $x = 4$. Find the volume of the solid obtained when this shaded region is rotated completely about the x-axis, giving your answer in terms of π. [4]

Answer: 42π. **J07/Q2**

16 The equation of a curve is $y = 2x + \dfrac{8}{x^2}$.

(i) Obtain expressions for $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$. [3]

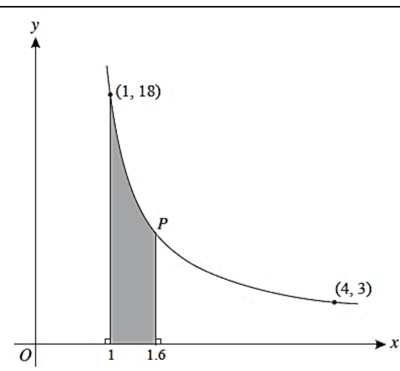
(ii) Find the coordinates of the stationary point on the curve and determine the nature of the stationary point. [3]

(iii) Show that the normal to the curve at the point $(-2, -2)$ intersects the x-axis at the point $(-10, 0)$. [3]

(iv) Find the area of the region enclosed by the curve, the x-axis and the lines $x = 1$ and $x = 2$. [3]

Answers: (i) $2 - \dfrac{16}{x^3}$, $\dfrac{48}{x^4}$; (ii) $(2, 6)$, Minimum; (iv) 7. **J07/Q10**

17



The diagram shows a curve for which $\frac{dy}{dx} = -\frac{k}{x^3}$, where k is a constant. The curve passes through the points (1, 18) and (4, 3).

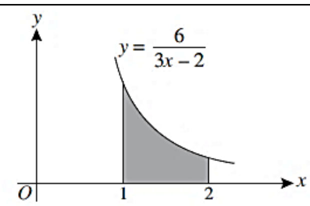
(i) Show, by integration, that the equation of the curve is $y = \frac{16}{x^2} + 2$. [4]

The point P lies on the curve and has x-coordinate 1.6.

(ii) Find the area of the shaded region. [4]

Answer: (ii) 7.2. **J08/Q9**

18



The diagram shows part of the curve $y = \frac{6}{3x-2}$.

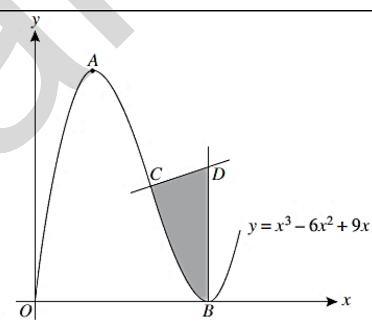
(i) Find the gradient of the curve at the point where $x = 2$. [3]

(ii) Find the volume obtained when the shaded region is rotated through 360° about the x-axis, giving your answer in terms of π. [5]

J09/Q9

Answers: (i) $-\frac{9}{8}$; (ii) 9π

19



The diagram shows the curve $y = x^3 - 6x^2 + 9x$ for $x \geqslant 0$. The curve has a maximum point at A and a minimum point on the x-axis at B. The normal to the curve at C (2, 2) meets the normal to the curve at B at the point D.

(i) Find the coordinates of A and B. [3]

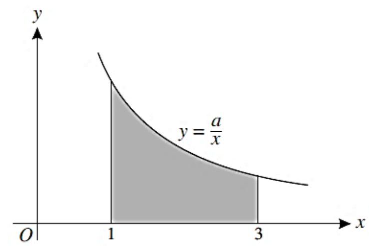
(ii) Find the equation of the normal to the curve at C. [3]

(iii) Find the area of the shaded region. [5]

J09/Q11

Answers: (i) A (1, 4), B (3, 0); (ii) $3y = x + 4$; (iii) $\frac{17}{12}$

20

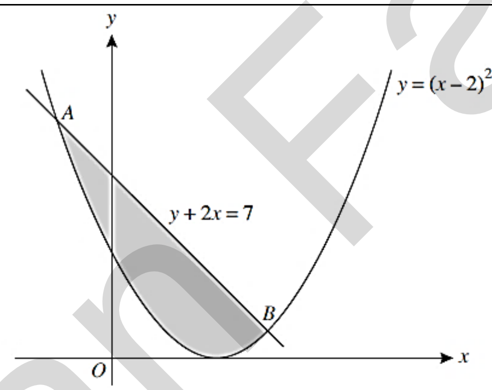


The diagram shows part of the curve $y = \frac{a}{x}$, where a is a positive constant. Given that the volume obtained when the shaded region is rotated through 360° about the x-axis is 24π, find the value of a. [4]

J10/12/Q2

Answer: 6.

21



The diagram shows the curve $y = (x-2)^2$ and the line $y + 2x = 7$, which intersect at points A and B. Find the area of the shaded region. [8]

J10/12/Q9

Answer: $10\frac{2}{3}$.

22

(a) Differentiate $4x + \frac{6}{x^2}$ with respect to x. [2]

(b) Find $\int \left(4x + \frac{6}{x^2}\right) dx$. [3]

J03/Q3

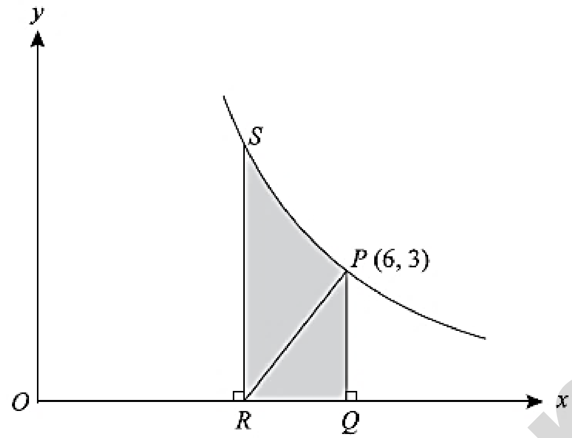
Answers: (a) $4 - \frac{12}{x^3}$; (b) $2x^2 - \frac{6}{x} + c$.

23

Evaluate $\int_0^1 \sqrt{(3x+1)}\, dx$. [4]

Answer: $1\frac{5}{9}$. J04/Q2

24



The diagram shows part of the graph of $y = \frac{18}{x}$ and the normal to the curve at P (6, 3). This normal meets the x-axis at R. The point Q on the x-axis and the point S on the curve are such that PQ and SR are parallel to the y-axis.

(i) Find the equation of the normal at P and show that R is the point $(4\frac{1}{2}, 0)$. [5]

(ii) Show that the volume of the solid obtained when the shaded region $PQRS$ is rotated through 360° about the x-axis is 18π. [4]

Answers: (i) $y = 2x - 9$; (ii) 18π. J04/Q7

25

A curve is such that $\frac{dy}{dx} = 2x^2 - 5$. Given that the point (3, 8) lies on the curve, find the equation of the curve. [4]

Answer: $y = \frac{2x^3}{3} - 5x + 5$. J05/Q1

26

A curve has equation $y = \frac{4}{\sqrt{x}}$.

(i) The normal to the curve at the point (4, 2) meets the x-axis at P and the y-axis at Q. Find the length of PQ, correct to 3 significant figures. [6]

(ii) Find the area of the region enclosed by the curve, the x-axis and the lines $x = 1$ and $x = 4$. [4]

Answers: (i) 14.4; (ii) 8 unit^2. J05/Q9

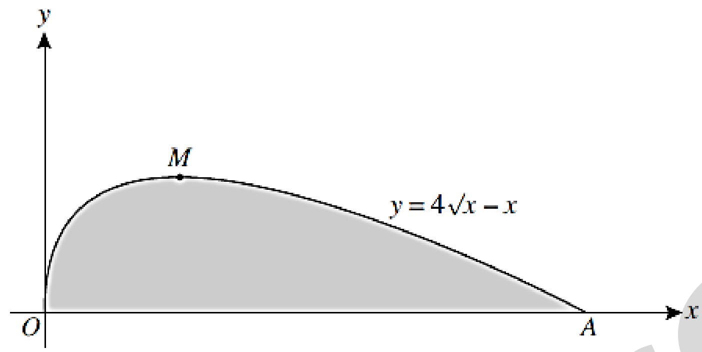
27

Find $\int \left(x^3 + \frac{1}{x^3}\right) dx$. [3]

Answer: $\frac{x^4}{4} - \frac{x^{-2}}{2} + c$.

J11/12/Q1

28



The diagram shows part of the curve $y = 4\sqrt{x} - x$. The curve has a maximum point at M and meets the x-axis at O and A.

(i) Find the coordinates of A and M. [5]

(ii) Find the volume obtained when the shaded region is rotated through 360° about the x-axis, giving your answer in terms of π. [6]

Answers: **(i)** (16, 0), (4, 4); **(ii)** 137π.

J11/12/Q11

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29 A curve is such that $\frac{dy}{dx} = 5 - \frac{8}{x^2}$. The line $3y + x = 17$ is the normal to the curve at the point P on the curve. Given that the x-coordinate of P is positive, find

(i) the coordinates of P, [4]

(ii) the equation of the curve. [4]

Answers: **(i)** (2, 5); **(ii)** $y = 5x + \frac{8}{x} - 9$

N11/12/Q7

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30 The equation of a curve is $y = \sqrt{(8x - x^2)}$. Find

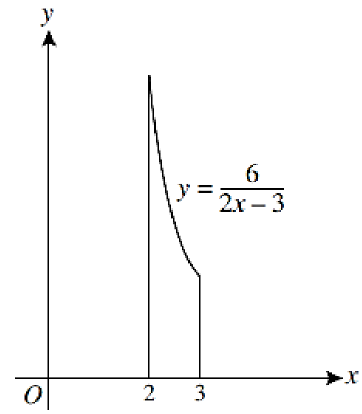
(i) an expression for $\frac{dy}{dx}$, and the coordinates of the stationary point on the curve, [4]

(ii) the volume obtained when the region bounded by the curve and the x-axis is rotated through 360° about the x-axis. [4]

Answers: **(i)** (4, 4); **(ii)** $\frac{256\pi}{3}$

N11/12/Q8

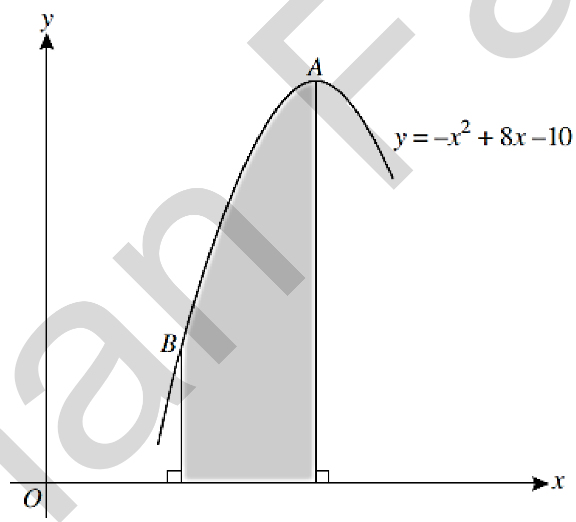
31



The diagram shows the region enclosed by the curve $y = \frac{6}{2x-3}$, the x-axis and the lines $x = 2$ and $x = 3$. Find, in terms of π, the volume obtained when this region is rotated through 360° about the x-axis. [4]

Answer: 12π. J12/12/Q1

32



The diagram shows part of the curve $y = -x^2 + 8x - 10$ which passes through the points A and B. The curve has a maximum point at A and the gradient of the line BA is 2.

(i) Find the coordinates of A and B. [7]

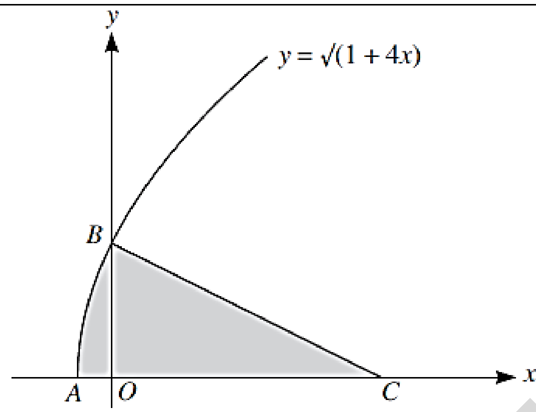
(ii) Find $\int y\,dx$ and hence evaluate the area of the shaded region. [4]

Answers: **(i)** A (4, 6), B (2, 2) **(ii)** $9\frac{1}{3}$ J12/12/Q9

33 A curve is such that $\frac{dy}{dx} = \frac{6}{x^2}$ and (2, 9) is a point on the curve. Find the equation of the curve. [3]

Answer: $y = -\frac{6}{x} + 12$. J13/12/Q1

34



The diagram shows the curve $y = \sqrt{(1 + 4x)}$, which intersects the x-axis at A and the y-axis at B. The normal to the curve at B meets the x-axis at C. Find

(i) the equation of BC, [5]

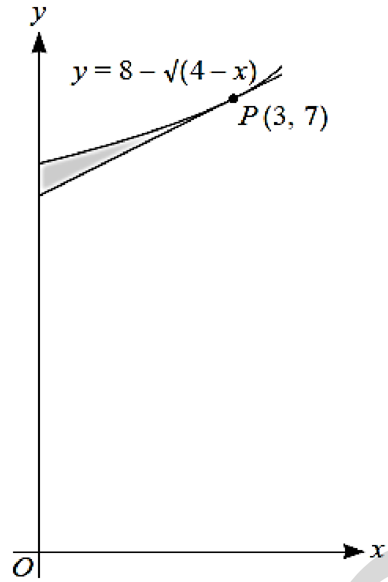
(ii) the area of the shaded region. [5]

Answers: **(i)** $2y + x = 2$. **(ii)** $1\frac{1}{6}$. J13/12/Q11

35 The equation of a curve is such that $\frac{d^2y}{dx^2} = 2x - 1$. Given that the curve has a minimum point at $(3, -10)$, find the coordinates of the maximum point. [8]

Answer: $(-2, 10\frac{5}{6})$ J14/12/Q8

36



The diagram shows part of the curve $y = 8 - \sqrt{(4 - x)}$ and the tangent to the curve at $P(3, 7)$.

(i) Find expressions for $\frac{dy}{dx}$ and $\int y\,dx$. [5]

(ii) Find the equation of the tangent to the curve at P in the form $y = mx + c$. [2]

(iii) Find, showing all necessary working, the area of the shaded region. [4]

Answers: **(i)** $\frac{1}{2\sqrt{4-x}}$, $8x + \frac{2}{3}(4-x)^{\frac{3}{2}}$ **(ii)** $y = \frac{1}{2}x + 5\frac{1}{2}$ **(iii)** $\frac{7}{12}$

J14/12/Q9

37 The function f is such that $f'(x) = 5 - 2x^2$ and $(3, 5)$ is a point on the curve $y = f(x)$. Find $f(x)$. [3]

Answer: $f(x) = 5x - \frac{2x^3}{3} + 8$

J15/12/Q1

38 The equation of a curve is $y = \frac{4}{2x - 1}$.

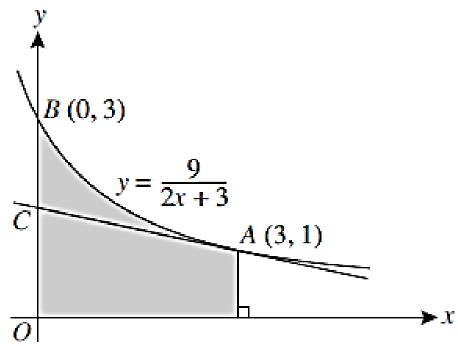
(i) Find, showing all necessary working, the volume obtained when the region bounded by the curve, the x-axis and the lines $x = 1$ and $x = 2$ is rotated through 360° about the x-axis. [4]

(ii) Given that the line $2y = x + c$ is a normal to the curve, find the possible values of the constant c. [6]

Answer: **(i)** $16\frac{\pi}{3}$; **(ii)** $\frac{5}{2}$ or $-\frac{7}{2}$

J15/12/Q10

39



The diagram shows part of the curve $y = \frac{9}{2x+3}$, crossing the y-axis at the point $B\,(0, 3)$. The point A on the curve has coordinates $(3, 1)$ and the tangent to the curve at A crosses the y-axis at C.

(i) Find the equation of the tangent to the curve at A. [4]

(ii) Determine, showing all necessary working, whether C is nearer to B or to O. [1]

(iii) Find, showing all necessary working, the exact volume obtained when the shaded region is rotated through 360° about the x-axis. [4]

Answers: **(i)** $9y + 2x = 15$; **(ii)** C is nearer to B; **(iii)** 9π. N12/12/Q9

40 A curve is defined for $x > 0$ and is such that $\frac{dy}{dx} = x + \frac{4}{x^2}$. The point $P\,(4, 8)$ lies on the curve.

(i) Find the equation of the curve. [4]

(ii) Show that the gradient of the curve has a minimum value when $x = 2$ and state this minimum value. [4]

Answers: **(i)** $y = \frac{x^2}{2} = \frac{4}{2} + 1$; **(ii)** Proof, Minimum value of the gradient = 3. N12/12/Q10

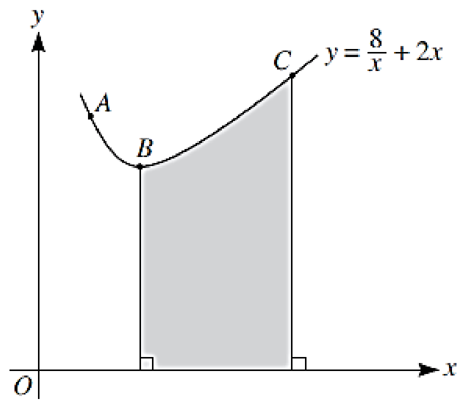
41 The equation of a curve is $y = \frac{2}{\sqrt{(5x-6)}}$.

(i) Find the gradient of the curve at the point where $x = 2$. [3]

(ii) Find $\int \frac{2}{\sqrt{(5x-6)}}\,dx$ and hence evaluate $\int_2^3 \frac{2}{\sqrt{(5x-6)}}\,dx$. [4]

Answers: **(i)** $-\frac{5}{8}$. **(ii)** $\frac{4}{5}\sqrt{5x-6}$, 0.8. N13/12/Q3

42



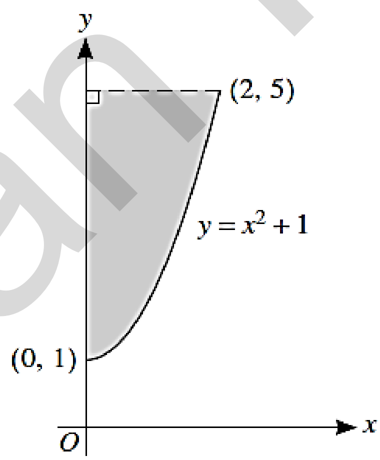
The diagram shows part of the curve $y = \frac{8}{x} + 2x$ and three points A, B and C on the curve with x-coordinates 1, 2 and 5 respectively.

(i) A point P moves along the curve in such a way that its x-coordinate increases at a constant rate of 0.04 units per second. Find the rate at which the y-coordinate of P is changing as P passes through A. [4]

(ii) Find the volume obtained when the shaded region is rotated through 360° about the x-axis. [6]

Answers: **(i)** -0.24. **(ii)** 271.2π or equivalent decimal answer. **N13/12/Q9**

43



The diagram shows part of the curve $y = x^2 + 1$. Find the volume obtained when the shaded region is rotated through 360° about the y-axis. [4]

Answer: 8π **N14/12/Q1**

44 A curve is such that $\frac{d^2y}{dx^2} = \frac{24}{x^3} - 4$. The curve has a stationary point at P where $x = 2$.

(i) State, with a reason, the nature of this stationary point. [1]

(ii) Find an expression for $\frac{dy}{dx}$. [4]

(iii) Given that the curve passes through the point (1, 13), find the coordinates of the stationary point P. [4]

Answers: **(ii)** $-12x^{-2} - 4x + 11$; **(iii)** (2, 12) **N14/12/Q10**

45 The curve $y = f(x)$ has a stationary point at (2, 10) and it is given that $f''(x) = \frac{12}{x^3}$.

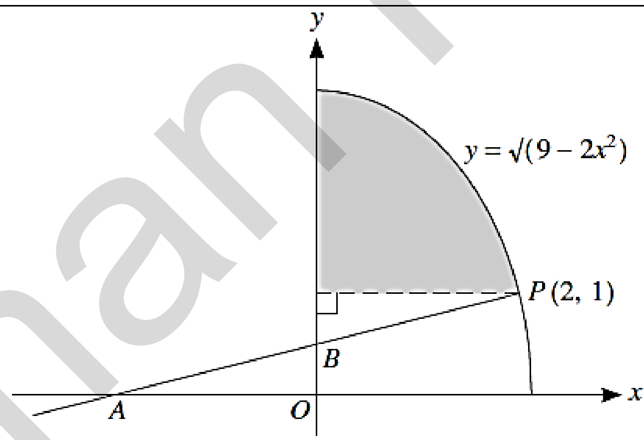
(i) Find $f(x)$. [6]

(ii) Find the coordinates of the other stationary point. [2]

(iii) Find the nature of each of the stationary points. [2]

Answers: **(i)** $f(x) = \frac{6}{x} + \frac{3x}{2} + 4$ **(ii)** (–2,–2) **(iii)** $x = 2$ *min*, $x = -2$ *max* **N15/12/Q9**

46



The diagram shows part of the curve $y = \sqrt{(9 - 2x^2)}$. The point $P(2, 1)$ lies on the curve and the normal to the curve at P intersects the x-axis at A and the y-axis at B.

(i) Show that B is the mid-point of AP. [6]

The shaded region is bounded by the curve, the y-axis and the line $y = 1$.

(ii) Find, showing all necessary working, the exact volume obtained when the shaded region is rotated through 360° about the y-axis. [5]

Answer: **(ii)** $\frac{14\pi}{3}$ **N15/12/Q10**

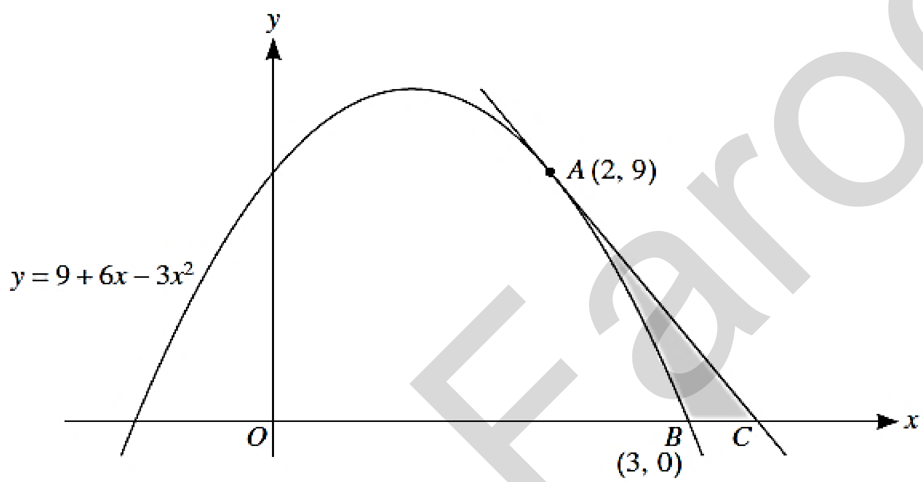
Homework: 9+Integration - Variants 11 & 13

1 A curve is such that $\frac{dy}{dx} = (2x+1)^{\frac{1}{2}}$ and the point (4, 7) lies on the curve. Find the equation of th curve. [4

Answer: $y = \frac{(2x+1)^{3/2}}{3} - 2$.

13/J15/2

2



Points A (2, 9) and B (3, 0) lie on the curve $y = 9 + 6x - 3x^2$, as shown in the diagram. The tangent at A intersects the x-axis at C. Showing all necessary working,

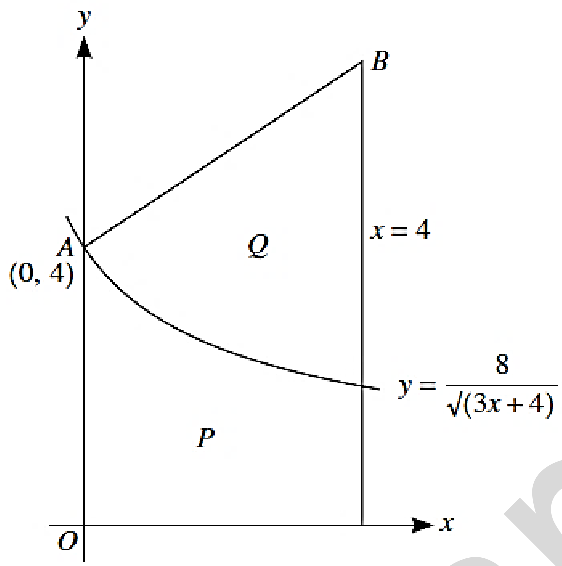
(i) find the equation of the tangent AC and hence find the x-coordinate of C, [4]

(ii) find the area of the shaded region ABC. [5]

Answers: (i) $y = -6x + 21$, $x = 3\frac{1}{2}$; (ii) $1\frac{3}{4}$

13/J15/10

3



The diagram shows part of the curve $y = \frac{8}{\sqrt{(3x+4)}}$. The curve intersects the y-axis at $A\ (0, 4)$. The normal to the curve at A intersects the line $x = 4$ at the point B.

(i) Find the coordinates of B. [5]

(ii) Show, with all necessary working, that the areas of the regions marked P and Q are equal. [6]

Answer: i) 4, 20/3 ii) 32/3

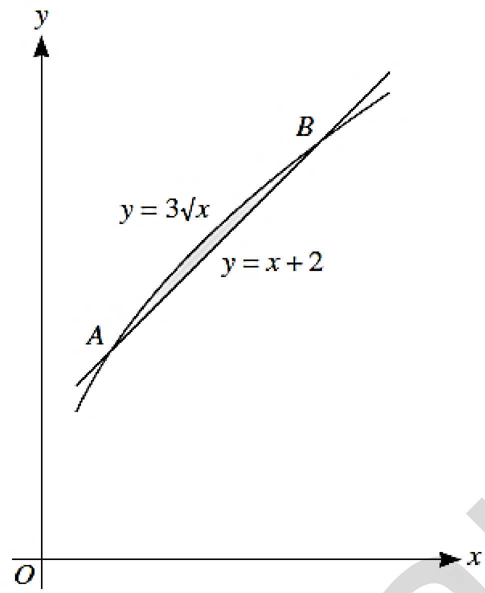
4 A curve $y = f(x)$ has a stationary point at $(3, 7)$ and is such that $f''(x) = 36x^{-3}$.

(i) State, with a reason, whether this stationary point is a maximum or a minimum. [1]

(ii) Find $f'(x)$ and $f(x)$. [7]

Answers: **(i)** Minimum since $f''(3) > 0$; **(ii)** $f(x) = 18x^{-1} + 2x - 5$ 13/N14/8

5



The diagram shows parts of the graphs of $y = x + 2$ and $y = 3\sqrt{x}$ intersecting at points A and B.

(i) Write down an equation satisfied by the x-coordinates of A and B. Solve this equation and hence find the coordinates of A and B. [4]

(ii) Find by integration the area of the shaded region. [6]

Answers: (i) (1, 3), (4, 6); (ii) $\frac{1}{2}$ 13/N14/9

6 The function f is defined for $x > 0$ and is such that $f'(x) = 2x - \frac{2}{x^2}$. The curve $y = f(x)$ passes through the point $P(2, 6)$.

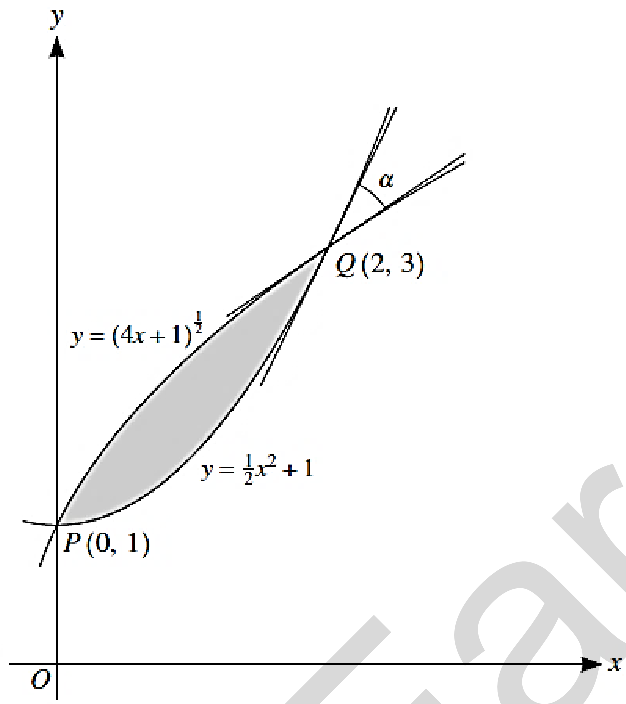
(i) Find the equation of the normal to the curve at P. [3]

(ii) Find the equation of the curve. [4]

(iii) Find the x-coordinate of the stationary point and state with a reason whether this point is a maximum or a minimum. [4]

Answers: (i) $7y + 2x = 46$ (ii) $y = x^2 + \frac{2}{x} + 1$ (iii) $x = 1$, minimum 11/N14/9

7



The diagram shows parts of the curves $y = (4x + 1)^{\frac{1}{2}}$ and $y = \frac{1}{2}x^2 + 1$ intersecting at points $P\,(0,\ 1)$ and $Q\,(2,\ 3)$. The angle between the tangents to the two curves at Q is α.

(i) Find α, giving your answer in degrees correct to 3 significant figures. [6]

(ii) Find by integration the area of the shaded region. [6]

Answers: **(i)** 29.7° **(ii)** 1

11/N14/11

8 A curve is such that $\frac{dy}{dx} = \frac{12}{\sqrt{(4x + a)}}$, where a is a constant. The point $P\,(2,\ 14)$ lies on the curve and the normal to the curve at P is $3y + x = 5$.

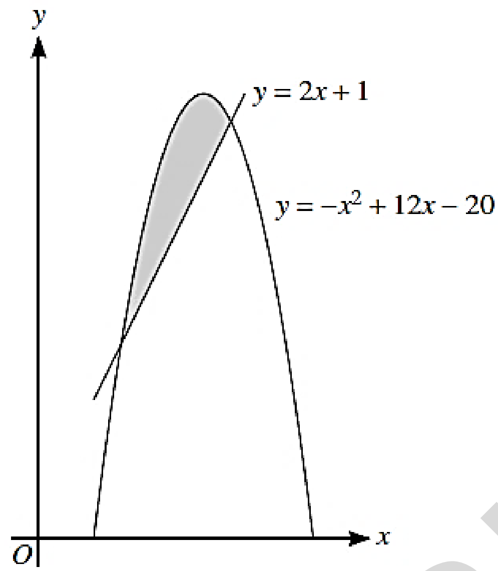
(i) Show that $a = 8$. [3]

(ii) Find the equation of the curve. [4]

Answer: **(ii)** $y = 6(4x + 8)^{\frac{1}{2}} - 10$.

13/J14/6

9



The diagram shows the curve $y = -x^2 + 12x - 20$ and the line $y = 2x + 1$. Find, showing all necessary working, the area of the shaded region. [8]

Answer: 10⅔. **13/J14/10**

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10 A line has equation $y = 2x + c$ and a curve has equation $y = 8 - 2x - x^2$.

(i) For the case where the line is a tangent to the curve, find the value of the constant c. [3]

(ii) For the case where $c = 11$, find the x-coordinates of the points of intersection of the line and the curve. Find also, by integration, the area of the region between the line and the curve. [7]

Answers: (i) $c = 12$; (ii) 1⅓. **11/J14/11**

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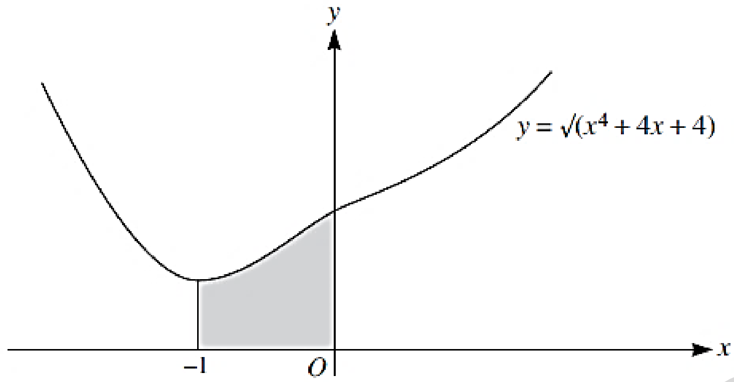
11 A curve has equation $y = f(x)$. It is given that $f'(x) = x^{-\frac{3}{2}} + 1$ and that $f(4) = 5$. Find $f(x)$. [4]

Answer: $f(x) = -2x^{-\frac{1}{2}} + x + 2$ **13/N13/2**

12



The diagram shows the curve $y = \sqrt{(x^4 + 4x + 4)}$.

(i) Find the equation of the tangent to the curve at the point (0, 2). [4]

(ii) Show that the x-coordinates of the points of intersection of the line $y = x + 2$ and the curve are given by the equation $(x + 2)^2 = x^4 + 4x + 4$. Hence find these x-coordinates. [4]

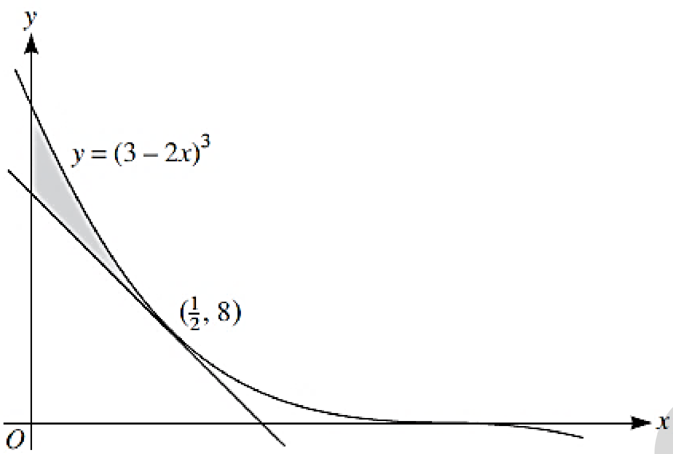
(iii) The region shaded in the diagram is rotated through 360° about the x-axis. Find the volume of revolution. [4]

Answer: $y - 2 = x$, 0,-1,1, $\frac{11\pi}{5}$ **13/N13/11**

13 A curve has equation $y = f(x)$. It is given that $f'(x) = \frac{1}{\sqrt{(x+6)}} + \frac{6}{x^2}$ and that $f(3) = 1$. Find $f(x)$. [5]

Answer: $2(x+6)^{1/2} - \frac{6}{x} - 3$. **11/N13/2**

14



The diagram shows the curve $y = (3 - 2x)^3$ and the tangent to the curve at the point $(\frac{1}{2}, 8)$.

(i) Find the equation of this tangent, giving your answer in the form $y = mx + c$. [5]

(ii) Find the area of the shaded region. [6]

Answers: **(i)** $y = -24x + 20$; **(ii)** 9/8 (or 1.125). 11/N13/10

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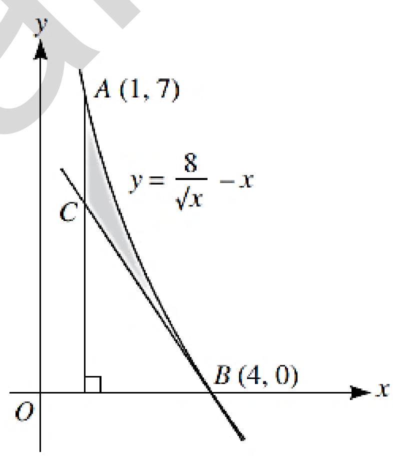
15 A curve is such that $\frac{dy}{dx} = \sqrt{(2x + 5)}$ and (2, 5) is a point on the curve. Find the equation of the curve. [4]

Answer: $y = ⅓\sqrt{(2x + 5)^3} - 4$. 13/J13/1

16



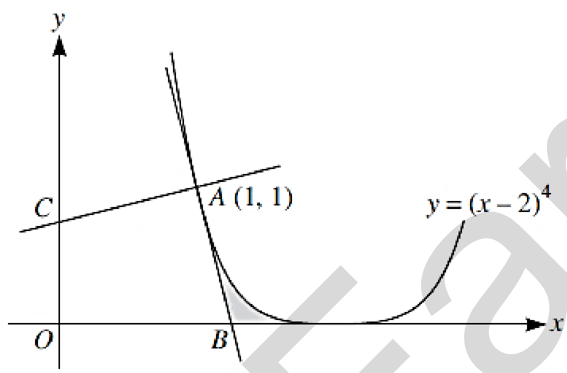
The diagram shows part of the curve $y = \dfrac{8}{\sqrt{x}} - x$ and points $A\,(1,\ 7)$ and $B\,(4,\ 0)$ which lie on the curve. The tangent to the curve at B intersects the line $x = 1$ at the point C.

(i) Find the coordinates of C. [4]

(ii) Find the area of the shaded region. [5]

Answers: **(i)** $(1, 4\tfrac{1}{2})$; **(ii)** $1\tfrac{3}{4}u^2$. 13/J13/11

17



The diagram shows part of the curve $y = (x - 2)^4$ and the point $A\,(1,\ 1)$ on the curve. The tangent at A cuts the x-axis at B and the normal at A cuts the y-axis at C.

(i) Find the coordinates of B and C. [6]

(ii) Find the distance AC, giving your answer in the form $\dfrac{\sqrt{a}}{b}$, where a and b are integers. [2]

(iii) Find the area of the shaded region. [4]

Answers: **(i)** $B(5/4, 0)$, $C(0, 3/4)$; **(ii)** $\sqrt{17}/4$; **(iii)** $3/40$. 11/J13/10

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18 A curve is such that

$$\frac{dy}{dx} = 2(3x+4)^{\frac{3}{2}} - 6x - 8.$$

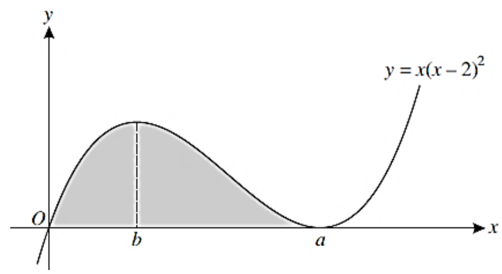
(i) Find $\dfrac{d^2y}{dx^2}$. [2]

(ii) Verify that the curve has a stationary point when $x = -1$ and determine its nature. [2]

(iii) It is now given that the stationary point on the curve has coordinates $(-1,\ 5)$. Find the equation of the curve. [5]

Answers: **(i)** $9(3x+4)^{\frac{1}{2}} - 6$; **(ii)** Minimum; **(iii)** $y = \dfrac{4}{15}(3x+4)^{\frac{5}{2}} - 3x^2 - 8x - \dfrac{4}{15}$. 13/N12/8

19



The diagram shows the curve with equation $y = x(x-2)^2$. The minimum point on the curve has coordinates $(a, 0)$ and the x-coordinate of the maximum point is b, where a and b are constants.

(i) State the value of a. [1]

(ii) Find the value of b. [4]

(iii) Find the area of the shaded region. [4]

(iv) The gradient, $\frac{dy}{dx}$, of the curve has a minimum value m. Find the value of m. [4]

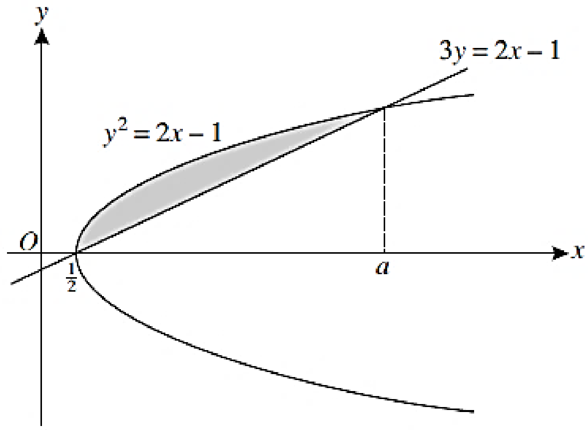
Answers: (i) 2; (ii) $\frac{2}{3}$; (iii) $\frac{4}{3}$; (iv) $\frac{-4}{3}$. 13/N12/11

---

20 A curve is such that $\frac{dy}{dx} = -\frac{8}{x^3} - 1$ and the point (2, 4) lies on the curve. Find the equation of the curve. [4]

Answer: $y = \frac{4}{x^2} - x + 5$. 11/N12/2

21



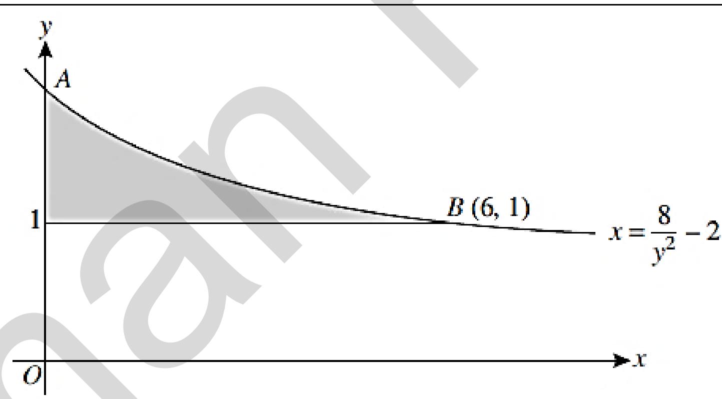
The diagram shows the curve $y^2 = 2x - 1$ and the straight line $3y = 2x - 1$. The curve and straight line intersect at $x = \frac{1}{2}$ and $x = a$, where a is a constant.

(i) Show that $a = 5$. [2]

(ii) Find, showing all necessary working, the area of the shaded region. [6]

Answer: (ii) 9/4. **11/N12/8**

22



The diagram shows part of the curve $x = \frac{8}{y^2} - 2$, crossing the y-axis at the point A. The point B (6, 1) lies on the curve. The shaded region is bounded by the curve, the y-axis and the line $y = 1$. Find the exact volume obtained when this shaded region is rotated through 360° about the y-axis. [6]

Answer: $6\frac{2}{3}\pi$ **13/J12/5**

23 A curve is such that $\frac{d^2y}{dx^2} = -4x$. The curve has a maximum point at (2, 12).

(i) Find the equation of the curve. [6]

A point P moves along the curve in such a way that the x-coordinate is increasing at 0.05 units per second.

(ii) Find the rate at which the y-coordinate is changing when $x = 3$, stating whether the y-coordinate is increasing or decreasing. [2]

13/J12/9

Answers: **(i)** $y = \frac{-2x^3}{3} + 8x + \frac{4}{3}$ **(ii)** $\frac{dy}{dt} = -0.5$ units per second, decreasing

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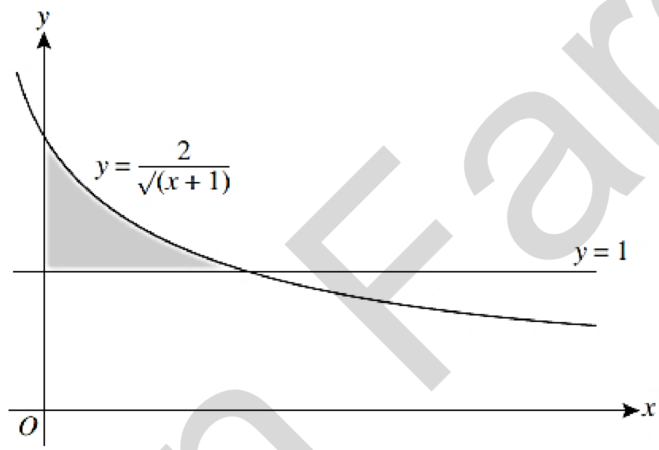
24



The diagram shows the line $y = 1$ and part of the curve $y = \frac{2}{\sqrt{(x+1)}}$.

(i) Show that the equation $y = \frac{2}{\sqrt{(x+1)}}$ can be written in the form $x = \frac{4}{y^2} - 1$. [1]

(ii) Find $\int \left(\frac{4}{y^2} - 1\right) dy$. Hence find the area of the shaded region. [5]

(iii) The shaded region is rotated through 360° about the y-axis. Find the exact value of the volume of revolution obtained. [5]

11/J12/11

Answers: **(ii)** $-\frac{4}{y} - y$, 1 **(iii)** $\frac{5\pi}{3}$

25 A curve $y = f(x)$ has a stationary point at $P(3, -10)$. It is given that $f'(x) = 2x^2 + kx - 12$, where k is a constant.

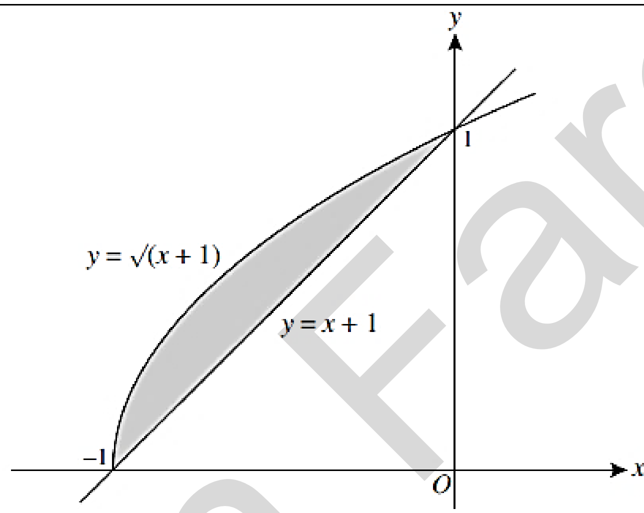
(i) Show that $k = -2$ and hence find the x-coordinate of the other stationary point, Q. [4]

(ii) Find $f''(x)$ and determine the nature of each of the stationary points P and Q. [2]

(iii) Find $f(x)$. [4]

Answers: (i) -2, (ii) min when $x = 3$ and max when $x = -2$, (iii) $f(x) = \frac{2}{3}x^3 - x^2 - 12x + 17$ 13/N11/8

26



The diagram shows the line $y = x + 1$ and the curve $y = \sqrt{(x + 1)}$, meeting at $(-1, 0)$ and $(0, 1)$.

(i) Find the area of the shaded region. [5]

(ii) Find the volume obtained when the shaded region is rotated through 360° about the y-axis. [7]

Answers: (i) $\frac{1}{6}$, (ii) $\frac{1}{5}\pi$ 13/N11/10

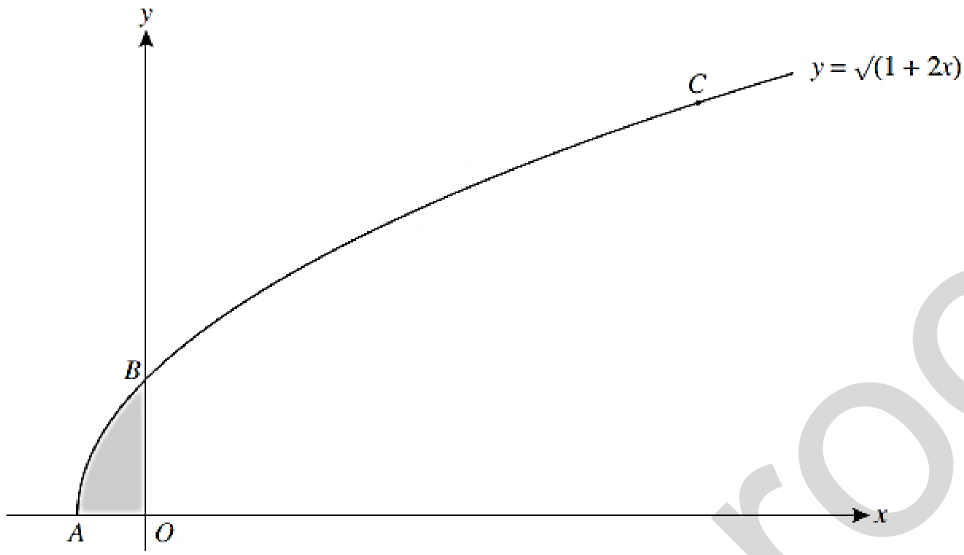
27 A function f is defined for $x \in \mathbb{R}$ and is such that $f'(x) = 2x - 6$. The range of the function is given by $f(x) \geqslant -4$.

(i) State the value of x for which $f(x)$ has a stationary value. [1]

(ii) Find an expression for $f(x)$ in terms of x. [4]

Answers: (i) 3; (ii) $x^2 - 6x + 5$. 11/N11/4

28



The diagram shows the curve $y = \sqrt{(1 + 2x)}$ meeting the x-axis at A and the y-axis at B. The y-coordinate of the point C on the curve is 3.

(i) Find the coordinates of B and C. [2]

(ii) Find the equation of the normal to the curve at C. [4]

(iii) Find the volume obtained when the shaded region is rotated through 360° about the y-axis. [5]

Answers: **(i)** (0, 1), (4, 3); **(ii)** $y = -3x + 15$; **(iii)** $\frac{2}{15}\pi$. 11/N11/10

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29 **(a)** Differentiate $\frac{2x^3 + 5}{x}$ with respect to x. [3]

(b) Find $\int (3x - 2)^5 \, dx$ and hence find the value of $\int_0^1 (3x - 2)^5 \, dx$. [4]

Answers: **(a)** $4x - 5x^{-2}$; **(b)** -3.5. 13/J11/4

---

30 A curve is such that $\frac{dy}{dx} = \frac{2}{\sqrt{x}} - 1$ and $P\,(9, 5)$ is a point on the curve.

(i) Find the equation of the curve. [4]

(ii) Find the coordinates of the stationary point on the curve. [3]

(iii) Find an expression for $\frac{d^2y}{dx^2}$ and determine the nature of the stationary point. [2]

(iv) The normal to the curve at P makes an angle of $\tan^{-1} k$ with the positive x-axis. Find the value of k. [2]

Answers: **(i)** $y = 4\sqrt{x} - x + 2$; **(ii)** (4, 6); **(iii)** $-x^{-3/2}$, Maximum; **(iv)** 3. 13/J11/9

31 **(i)** Sketch the curve $y = (x-2)^2$. [1]

(ii) The region enclosed by the curve, the x-axis and the y-axis is rotated through 360° about the x-axis. Find the volume obtained, giving your answer in terms of π. [4]

Answer: **(ii)** $\frac{32\pi}{5}$. 11/J11/3

32 A curve is such that $\frac{dy}{dx} = \frac{3}{(1+2x)^2}$ and the point $(1, \frac{1}{2})$ lies on the curve.

(i) Find the equation of the curve. [4]

(ii) Find the set of values of x for which the gradient of the curve is less than $\frac{1}{3}$. [3]

Answers: **(i)** $y = \frac{-3}{2(1+2x)} + 1$; **(ii)** $x < -2$, $x > 1$. 11/J11/7

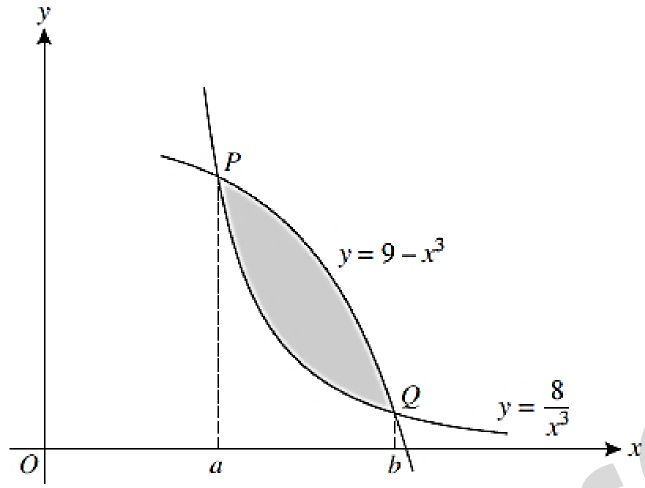
33 A curve has equation $y = f(x)$. It is given that $f'(x) = 3x^2 + 2x - 5$.

(i) Find the set of values of x for which f is an increasing function. [3]

(ii) Given that the curve passes through (1, 3), find $f(x)$. [4]

Answers: **(i)** $x < -\frac{5}{3}$, $x > 1$; **(ii)** $x^3 + x^2 - 5x + 6$. 13/N10/6

34



The diagram shows parts of the curves $y = 9 - x^3$ and $y = \frac{8}{x^3}$ and their points of intersection P and Q. The x-coordinates of P and Q are a and b respectively.

(i) Show that $x = a$ and $x = b$ are roots of the equation $x^6 - 9x^3 + 8 = 0$. Solve this equation and hence state the value of a and the value of b. [4]

(ii) Find the area of the shaded region between the two curves. [5]

(iii) The tangents to the two curves at $x = c$ (where $a < c < b$) are parallel to each other. Find the value of c. [4]

Answers: (i) 1, 2; (ii) 2.25; (iii) $\sqrt{2}$. 13/N10/11

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35 Find $\int \left(x + \frac{1}{x}\right)^2 dx$. [3]

Answer. $\frac{x^3}{3} - \frac{1}{x} + 2x + c$. 11/N10/1

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36 The equation of a curve is $y = \frac{9}{2 - x}$.

(i) Find an expression for $\frac{dy}{dx}$ and determine, with a reason, whether the curve has any stationary points. [3]

(ii) Find the volume obtained when the region bounded by the curve, the coordinate axes and the line $x = 1$ is rotated through $360°$ about the x-axis. [4]

(iii) Find the set of values of k for which the line $y = x + k$ intersects the curve at two distinct points. [4]

11/N10/11

Answers: **(i)** $\frac{9}{(2-x)^2}$; **(ii)** $\frac{81\pi}{2}$; **(iii)** $k < -8$, $k > 4$.

37 The equation of a curve is such that $\frac{dy}{dx} = \frac{6}{\sqrt{(3x-2)}}$. Given that the curve passes through the point $P(2, 11)$, find

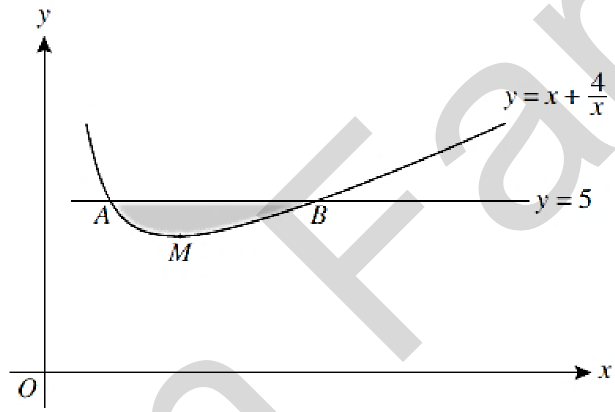
(i) the equation of the normal to the curve at P, [3]

(ii) the equation of the curve. [4]

13/J10/5

Answers: **(i)** $3y + x = 35$; **(ii)** $y = 4\sqrt{3x-2} + 3$.

38



The diagram shows part of the curve $y = x + \frac{4}{x}$ which has a minimum point at M. The line $y = 5$ intersects the curve at the points A and B.

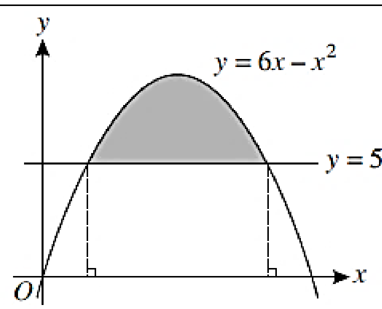
(i) Find the coordinates of A, B and M. [5]

(ii) Find the volume obtained when the shaded region is rotated through 360° about the x-axis. [6]

13/J10/9

Answers: **(i)** (1, 5), (4, 5), (2, 4); **(ii)** 56.5 or 18π.

39



The diagram shows the curve $y = 6x - x^2$ and the line $y = 5$. Find the area of the shaded region. [6]

11/J10/4

Answer: $10\frac{2}{3}$.

---

40 A curve is such that $\frac{dy}{dx} = 3x^{\frac{1}{2}} - 6$ and the point $(9, 2)$ lies on the curve.

(i) Find the equation of the curve. [4]

(ii) Find the x-coordinate of the stationary point on the curve and determine the nature of the stationary point. [3]

11/J10/6

Answers: (i) $y = 2x^{\frac{3}{2}} - 6x + 2$; (ii) 4, Minimum.

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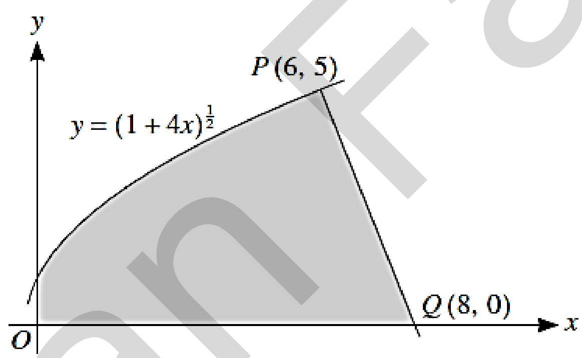
41 The function f is such that $f'(x) = 3x^2 - 7$ and $f(3) = 5$. Find $f(x)$. [3]

N15/11/Q2

42



The diagram shows part of the curve $y = (1 + 4x)^{\frac{1}{2}}$ and a point $P(6, 5)$ lying on the curve. The line PQ intersects the x-axis at $Q(8, 0)$.

(i) Show that PQ is a normal to the curve. [5]

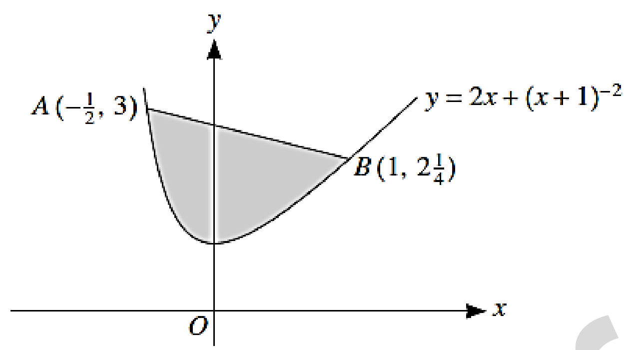
(ii) Find, showing all necessary working, the exact volume of revolution obtained when the shaded region is rotated through 360° about the x-axis. [7]

[In part **(ii)** you may find it useful to apply the fact that the volume, V, of a cone of base radius r and vertical height h, is given by $V = \frac{1}{3}\pi r^2 h$.]

N15/11/Q11

43 The function f is defined by $f(x) = 2x + (x+1)^{-2}$ for $x > -1$.

(i) Find $f'(x)$ and $f''(x)$ and hence verify that the function f has a minimum value at $x = 0$. [4]



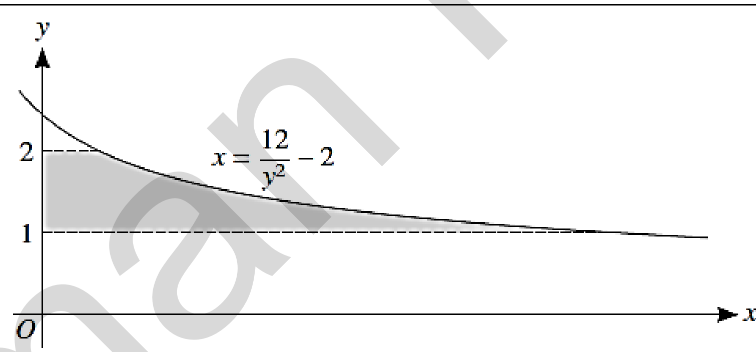
The points $A(-\frac{1}{2}, 3)$ and $B(1, 2\frac{1}{4})$ lie on the curve $y = 2x + (x+1)^{-2}$, as shown in the diagram.

(ii) Find the distance AB. [2]

(iii) Find, showing all necessary working, the area of the shaded region. [6]

N15/13/Q10

44



The diagram shows part of the curve $x = \frac{12}{y^2} - 2$. The shaded region is bounded by the curve, the y-axis and the lines $y = 1$ and $y = 2$. Showing all necessary working, find the volume, in terms of π, when this shaded region is rotated through 360° about the y-axis. [5]

Answer: 22π

J16/11/Q3

45 A curve is such that $\frac{\mathrm{d}y}{\mathrm{d}x} = 2 - 8(3x+4)^{-\frac{1}{2}}$.

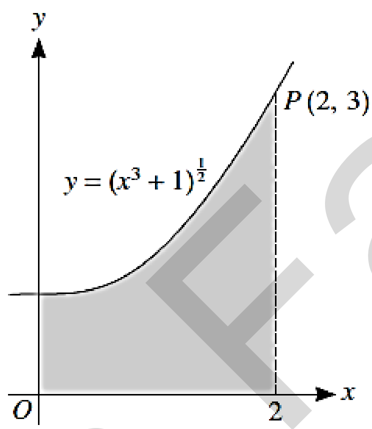
(i) A point P moves along the curve in such a way that the x-coordinate is increasing at a constant rate of 0.3 units per second. Find the rate of change of the y-coordinate as P crosses the y-axis. [2]

The curve intersects the y-axis where $y = \frac{4}{3}$.

(ii) Find the equation of the curve. [4]

Answers: (i) –0.6 (ii) $y = 2x - \frac{16}{3}\sqrt{3x+4} + 12$ J16/11/Q4

46



The diagram shows part of the curve $y = (x^3+1)^{\frac{1}{2}}$ and the point $P(2, 3)$ lying on the curve. Find, showing all necessary working, the volume obtained when the shaded region is rotated through 360° about the x-axis. [4]

Answer: 6π. J16/13/Q2

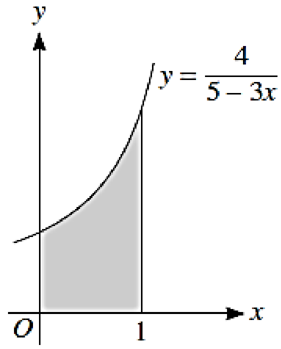
47 A curve is such that $\frac{\mathrm{d}y}{\mathrm{d}x} = 6x^2 + \frac{k}{x^3}$ and passes through the point $P(1, 9)$. The gradient of the curve at P is 2.

(i) Find the value of the constant k. [1]

(ii) Find the equation of the curve. [4]

Answers: (i) $k = 4$; (ii) $y = 2x^3 + 2x^{-2} + 5$. J16/13/Q3

48



The diagram shows part of the curve $y = \dfrac{4}{5 - 3x}$.

(i) Find the equation of the normal to the curve at the point where $x = 1$ in the form $y = mx + c$, where m and c are constants. [5]

The shaded region is bounded by the curve, the coordinate axes and the line $x = 1$.

(ii) Find, showing all necessary working, the volume obtained when this shaded region is rotated through 360° about the x-axis. [5]

J17/11/Q10

Answers: **(i)** $y = -\dfrac{1}{3}x + \dfrac{7}{3}$ **(ii)** $\dfrac{8\pi}{5}$ or equivalent

49 **(a)**

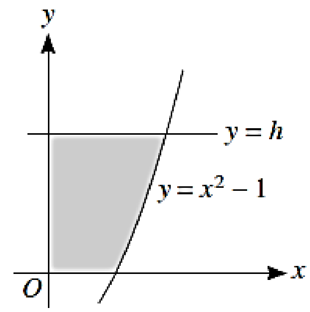


Fig. 1

Fig. 1 shows part of the curve $y = x^2 - 1$ and the line $y = h$, where h is a constant.

(i) The shaded region is rotated through 360° about the y-axis. Show that the volume of revolution, V, is given by $V = \pi\left(\frac{1}{2}h^2 + h\right)$. [3]

(ii) Find, showing all necessary working, the area of the shaded region when $h = 3$. [4]

(b)

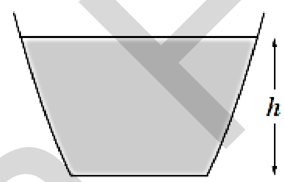


Fig. 2

Fig. 2 shows a cross-section of a bowl containing water. When the height of the water level is h cm, the volume, V cm^3, of water is given by $V = \pi\left(\frac{1}{2}h^2 + h\right)$. Water is poured into the bowl at a constant rate of 2 cm^3 s^{-1}. Find the rate, in cm s^{-1}, at which the height of the water level is increasing when the height of the water level is 3 cm. [4]

Answers : **(a)(ii)** 14 / 3; **(b)** 1 / (2π) or 0.159. **J17/13/Q10**

50 The function f is defined for $x \geqslant 0$. It is given that f has a minimum value when $x = 2$ and that $f''(x) = (4x+1)^{-\frac{1}{2}}$.

(i) Find $f'(x)$. [3]

It is now given that $f''(0)$, $f'(0)$ and $f(0)$ are the first three terms respectively of an arithmetic progression.

(ii) Find the value of $f(0)$. [3]

19

(iii) Find $f(x)$, and hence find the minimum value of f. [5]

Answers : **(i)** $\frac{1}{2}(4x+1)^{1/2} - \frac{3}{2}$; **(ii)** f(0) = –3; **(iii)** minimum value = $-\frac{23}{6}$. **J17/13/Q11**

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51 A curve is such that $\frac{dy}{dx} = \frac{12}{(2x+1)^2}$. The point (1, 1) lies on the curve. Find the coordinates of the point at which the curve intersects the x-axis. [6]

Answer: $(\frac{1}{2}, 0)$ **J18/11/Q3**

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52 The curve with equation $y = x^3 - 2x^2 + 5x$ passes through the origin.

(i) Show that the curve has no stationary points. [3]

(ii) Denoting the gradient of the curve by m, find the stationary value of m and determine its nature. [5]

(iii) Showing all necessary working, find the area of the region enclosed by the curve, the x-axis and the line $x = 6$. [4]

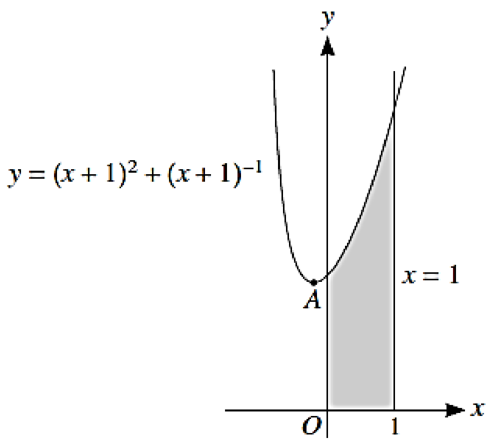
Answers: **(ii)** $m = \frac{11}{3}$, minimum **(iii)** 270 **J18/11/Q10**

---

53 A curve with equation $y = f(x)$ passes through the point A (3, 1) and crosses the y-axis at B. It is given that $f'(x) = (3x-1)^{-\frac{1}{3}}$. Find the y-coordinate of B. [6]

Answer: $-\frac{1}{2}$. **J18/13/Q4**

54



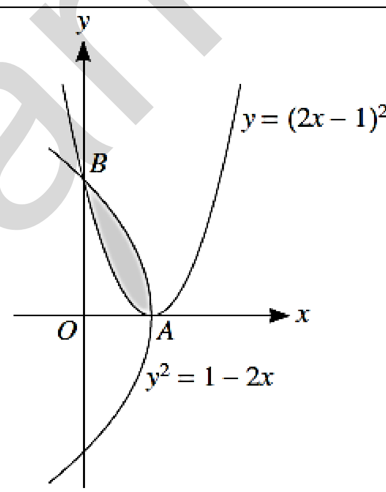
The diagram shows part of the curve $y = (x+1)^2 + (x+1)^{-1}$ and the line $x = 1$. The point A is the minimum point on the curve.

(i) Show that the x-coordinate of A satisfies the equation $2(x+1)^3 = 1$ and find the exact value of $\frac{d^2y}{dx^2}$ at A. [5]

(ii) Find, showing all necessary working, the volume obtained when the shaded region is rotated through 360° about the x-axis. [6]

Answers: **(i)** $\frac{d^2y}{dx^2} = 6$; **(ii)** 9.7π. J18/13/Q11

55



The diagram shows parts of the curves $y = (2x-1)^2$ and $y^2 = 1 - 2x$, intersecting at points A and B.

(i) State the coordinates of A. [1]

(ii) Find, showing all necessary working, the area of the shaded region. [6]

Answers: **(i)** (1/2,0) **(ii)** 1/6

N16/11/Q7

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56 A curve has equation $y = f(x)$ and it is given that $f'(x) = 3x^{\frac{1}{2}} - 2x^{-\frac{1}{2}}$. The point A is the only point on the curve at which the gradient is -1.

(i) Find the x-coordinate of A. [3]

(ii) Given that the curve also passes through the point $(4, 10)$, find the y-coordinate of A, giving your answer as a fraction. [6]

Answers: **(i)** 4/9 **(ii)** –2/27

N16/11/Q10

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57 The function f is such that $f(x) = x^3 - 3x^2 - 9x + 2$ for $x > n$, where n is an integer. It is given that f is an increasing function. Find the least possible value of n. [4]

Answer: The least possible value of n is 3.

N16/13/Q4

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58 A curve is such that $\frac{dy}{dx} = \frac{2}{a}x^{-\frac{1}{2}} + ax^{-\frac{3}{2}}$, where a is a positive constant. The point $A\,(a^2, 3)$ lies on the curve. Find, in terms of a,

(i) the equation of the tangent to the curve at A, simplifying your answer, [3]

(ii) the equation of the curve. [4]

It is now given that $B\,(16, 8)$ also lies on the curve.

(iii) Find the value of a and, using this value, find the distance AB. [5]

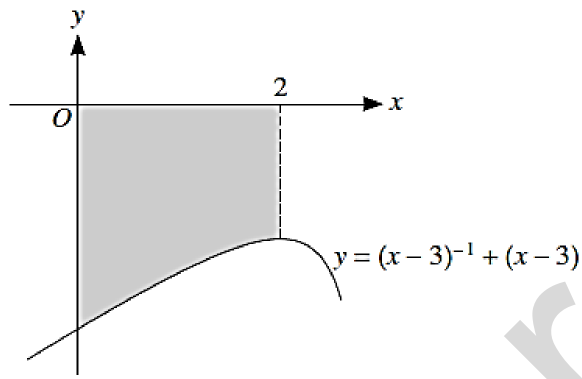
Answers: **(i)** $y = \frac{3}{a^2}x$; **(ii)** $y = \frac{4x^{½}}{a} - 2ax^{-½} + 1$; **(iii)** $a = 2$, $AB = 13$

N16/13/Q10

59 A curve has equation $y = (kx - 3)^{-1} + (kx - 3)$, where k is a non-zero constant.

(i) Find the x-coordinates of the stationary points in terms of k, and determine the nature of each stationary point, justifying your answers. [7]

(ii)

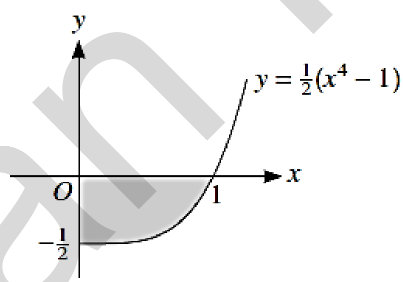


The diagram shows part of the curve for the case when $k = 1$. Showing all necessary working, find the volume obtained when the region between the curve, the x-axis, the y-axis and the line $x = 2$, shown shaded in the diagram, is rotated through 360° about the x-axis. [5]

Answers: **(i)** $x = 2/k$ maximum, $x = 4/k$ minimum; **(ii)** $40\pi/3$.

N16/13/Q11

60



The diagram shows part of the curve $y = \frac{1}{2}(x^4 - 1)$, defined for $x \geqslant 0$.

(i) Find, showing all necessary working, the area of the shaded region. [3]

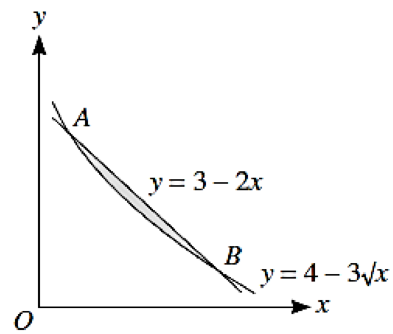
(ii) Find, showing all necessary working, the volume obtained when the shaded region is rotated through 360° about the x-axis. [4]

(iii) Find, showing all necessary working, the volume obtained when the shaded region is rotated through 360° about the y-axis. [5]

Answers: **(i)** $\frac{2}{5}$ **(ii)** $\frac{8\pi}{45}$ **(iii)** $\frac{\pi}{3}$

N17/11/Q10

61



The diagram shows parts of the graphs of $y = 3 - 2x$ and $y = 4 - 3\sqrt{x}$ intersecting at points A and B.

(i) Find by calculation the x-coordinates of A and B. [3]

(ii) Find, showing all necessary working, the area of the shaded region. [5]

Answers: **(i)** $x = \frac{1}{4}, 1$ **(ii)** $\frac{1}{16}$ — N17/13/Q8

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62 A curve has equation $y = f(x)$ and it is given that $f'(x) = ax^2 + bx$, where a and b are positive constants.

(i) Find, in terms of a and b, the non-zero value of x for which the curve has a stationary point and determine, showing all necessary working, the nature of the stationary point. [3]

17

(ii) It is now given that the curve has a stationary point at $(-2, -3)$ and that the gradient of the curve at $x = 1$ is 9. Find $f(x)$. [6]

Answers: **(i)** $x = -\frac{b}{a}$, Maximum **(ii)** $f(x) = x^3 + 3x^2 - 7$ — N17/13/Q10

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63 A curve has a stationary point at $(3, 9\frac{1}{2})$ and has an equation for which $\frac{dy}{dx} = ax^2 + a^2x$, where a is a non-zero constant.

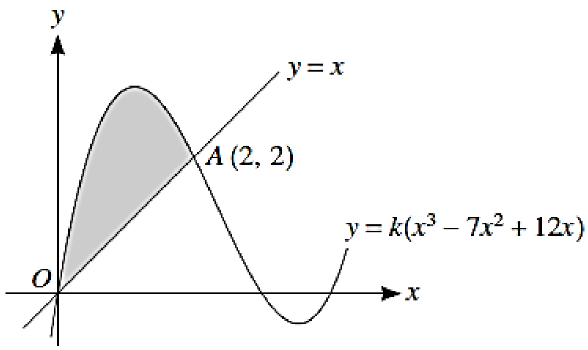
(i) Find the value of a. [2]

(ii) Find the equation of the curve. [4]

(iii) Determine, showing all necessary working, the nature of the stationary point. [2]

Answers: **(i)** -3 **(ii)** $y = -x^3 + \frac{9x^2}{2} - 4$ **(iii)** maximum — N18/11/Q6

64



The diagram shows part of the curve with equation $y = k(x^3 - 7x^2 + 12x)$ for some constant k. The curve intersects the line $y = x$ at the origin O and at the point A (2, 2).

(i) Find the value of k. [1]

(ii) Verify that the curve meets the line $y = x$ again when $x = 5$. [2]

(iii) Find, showing all necessary working, the area of the shaded region. [5]

Answers: **(i)** $\frac{1}{2}$ **(iii)** $\frac{8}{3}$ **N18/11/Q7**

65

A curve passes through (0, 11) and has an equation for which $\frac{dy}{dx} = ax^2 + bx - 4$, where a and b are constants.

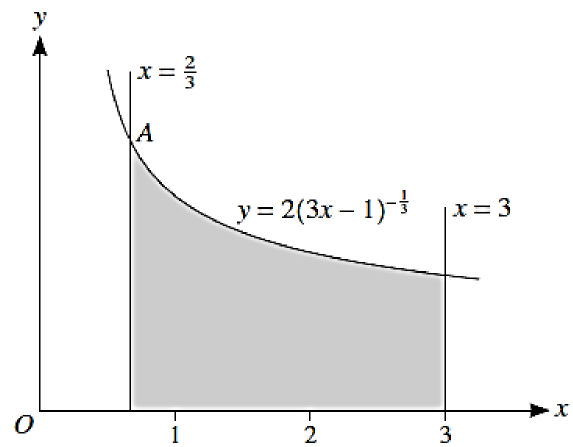
(i) Find the equation of the curve in terms of a and b. [3]

13

(ii) It is now given that the curve has a stationary point at (2, 3). Find the values of a and b. [5]

Answers: **(i)** $y = \frac{1}{3}ax^3 + \frac{1}{2}bx^2 - 4x + 11$ **(ii)** $a = 3$ $b = -4$ **N18/13/Q8**

66



The diagram shows part of the curve $y = 2(3x-1)^{-\frac{1}{3}}$ and the lines $x = \frac{2}{3}$ and $x = 3$. The curve and the line $x = \frac{2}{3}$ intersect at the point A.

(i) Find, showing all necessary working, the volume obtained when the shaded region is rotated through 360° about the x-axis. [5]

(ii) Find the equation of the normal to the curve at A, giving your answer in the form $y = mx + c$. [5]

Answers: **(i)** 4π **(ii)** $y = \left(\frac{1}{2}\right)x + \frac{5}{3}$

N18/13/Q10